Misspecified MA(2) Model Fitting to a Data from Gaussian MA(q) Process

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Abstract

This paper provides discussions about a misspecified MA(2) model fitting to (a data generated by) a Gaussian MA(q) process. They are mainly concerned a problem for finding a number of locally maximal points of the Gaussian quasi-maximum likelihood function of the model when the sample size is large. When $0 \leq q \leq 3$, the MA(2) model has always only one parameter set estimated in the invertible parameter space. On the other hand when $q \geq 5$, the likelihood function of the MA(2) model has more than one locally maximal points in the invertible parameter space if the model is fitted to a data from some MA(q) process.

Key words: *MA(q) model fitting; conditional maximum likelihood function; Gaussian; locally minimal points; misspecification.*

1. Introduction.

It is well known that when we fit an MA(1) model to some special time series data which does not follow MA(1) process, the MA(1) parameter does not always have an unique Gaussian quasi-maximum likelihood estimator in the invertible space. Tanaka and Huzii [11] have given the conditions of AR(2) parameters on which the MA(1) quasi-likelihood function has more than one locally maximal points in the invertible parameter space $(-1, 1)$. Tanaka and Aoki [10] also showed the domain for the AR(2) parameters on which the MA(1) conditional-likelihood function has more than one locally maximal points in the AR(2) parameter space shown in Figure 1 below. From Tanaka and Huzii [11], we have two locally maximal points of the MA(1) conditional-likelihood function $F(x;a,b)=F(x)$, say, where x is an MA(1) parameter and a, b are AR(2) parameters. In order to have the conditions on which the function has two locally maximal points in the AR-parameter space, we should consider the differentiation $DF(x) = 0$, and we specified the case where the solution of the equation $DF(x) = 0$ changed from three to two. That is, the value of the resultant (see [8], [9]) was able to formalize the contour line for zero (the bifurcation set). We set the domain with a deep color surrounded with the curve of the shape of a wedge given in the upper part of Figure 1. Its boundary is the bifurcation set. It will be seen that the function $F(x;a,b)$ is locally a cusp (see [4], [7], [10]).

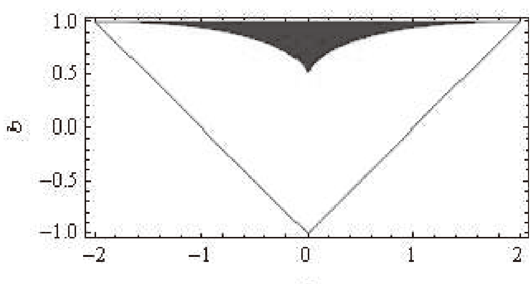

Figure 1. The domain for AR(2)-parameters.

In this paper, we shall consider an MA(2) model fitting to the data from MA(q) processes. To know answers of problems some simulation studies and some numerical integrations are performed by using Mathematica software (*Ver.11.3*).

From our findings, we shall conjecture that there are more than one locally maximal points of the conditional likelihood function in the invertible parameter space, if an MA(2) model is fitted to some series belongs to an MA(q) process for the order $q \geq 5$.

2. Notations for an MA(2) model fitting.

Let $\{Z(t)\}$ be a weakly stationary process with $EZ(t) = 0$. $\{Z(t)\}$ is said to satisfy a moving average model of order q (MA(q) model) if $\{Z(t)\}$ is expressed as

$$Z(t) = (1 + \beta_1 B + \ldots + \beta_q B^q)\, \mathrm{e(t)}, \tag{2.1}$$

where $\{e(t)\}$, t being an integer, consists of independently and identically distributed random variables with $E[e(t)] = 0$, $E\left[e(t)^2\right] = \sigma^2$, the β_q's are constants which are independent of t, and B is the usual back-shift operator such that $B[e(t)] = e(t-1)$ and $B^k[e(t)] = B\left[B^{k-1}[e(t)]\right]$ for $k = 1,2,..$ (see, for example [1], [2], [3]).

A function $\theta(B)$ is given by

$$\theta(B) = 1 + \beta_1 B + \ldots + \beta_q B^q = \prod_{k=1}^{q} (1 - \theta_k B). \tag{2.2}$$

In our model fitting, it is assumed that $|\theta_k| \leq 1$ for all $k = 1, 2, \cdots, q$. Let $\Theta = (\beta_1, \ldots, \beta_q)$ be a q-dimensional unknown parameter, and let $\{F_k(\Theta)\}$ be a sequence of functions of Θ, which are defined in the following way. For $t > 0$,

$$e(t) = \prod_{k=1}^{q} (1 - \theta_k B)^{-1}\, \mathrm{Z(t)} = \left\{\sum_{k=1}^{\infty} F_k(\Theta)\, B^k\right\} Z(t). \tag{2.3}$$

For evaluating the asymptotic properties of the conditional quasi-maximum likelihood estimators of Θ when the sample size tends to infinity, we should attend to a function

$$S_q(\Theta) = E\left[e(t)^2\right] \tag{2.4}$$

$$= \int_{-1/2}^{1/2} \frac{1}{\left|\prod_{j=1}^{q}\left[1 - \theta_j \exp(-2\pi \mathrm{i}\omega)\right]\right|^2} f_Z(\omega)\, d\omega.$$

The value $\hat{\Theta}$ which minimizes $S_q(\Theta)$ with respect to Θ should be obtained for the conditional quasi-maximum likelihood estimators of Θ (see Tanaka and Huzii [11], Huzii [5], Kabaila [6]).

The spectrum of an MA(q) process $f_Z(\omega)$ is given by

$$f_Z(\omega) = \frac{\sigma^2}{2\pi} \left|\theta\left(e^{-i\omega}\right)\right|^2. \tag{2.5}$$

If the process $\{Z(t)\}$ is an MA(q) process and it is correctly fitted by the MA(q) model, then we have $S_q(\Theta) = \frac{\sigma^2}{2\pi}$, which is a spectral density of a white noise process.

Now let $\{X(t)\}$ be a weakly stationary process with mean $E[X(t)] = 0$ and spectral density $f_X(\omega)$. For an MA(q) model fitting to this process $\{X(t)\}$, $S_q(\Theta)$ is expressed as

$$S_q(\Theta) = \int_{-1/2}^{1/2} \frac{f_X(\omega)}{\left|\prod_{j=1}^{q}\left[1-\theta_j \exp(-2\pi i\omega)\right]\right|^2} d\omega. \tag{2.6}$$

In this paper, our consideration is given to the case when an MA(2) model is fitted incorrectly to an MA(q) process $\{X(t)\}$; $X(t) = (1 + b_1 B + \ldots + b_q B^q)\, e(t)$.

Then we set the MA(2) model parameters (x, y) in stead of (β_1, β_2), and define a function

$$S_{2,q}(\Theta) = S_2(\Theta).$$

For $q = 2$, $S_{2,2}(\Theta)$ can be derived from (2.6), ignoring the constant term $\frac{\sigma^2}{2\pi}$, as

$$S_{2,2}(x,\ y) = S_{2,2}(x,\ y\ ;\ b_1,\ b_2)$$

$$= \frac{1+y+(1+y)\, b_1^2+2\left(x^2-y\,(1+y)\right) b_2+(1+y)\, b_2^2-2\, x\, b_1\,(1+b_2)}{(1-y)\,(1-x+y)\,(1+x+y)}. \tag{2.7}$$

The invertible parameter space Ω_2 of the MA(2) model is given by

$$\Omega_2 = \{(x, y)\ ;\ 1-y^2>0,\ (x-x\,y)\,(-x+x\,y)+\left(1-y^2\right)^2>0\}. \tag{2.8}$$

If we fit the MA(2) model to a special MA(5) process, the function $S_{2,5}(x,\ y)$ will have two locally minimal points in an invertible space Ω_2 of the MA(2) model. For an example, we have the following graph for an MA(5) process with $(b_1, \ldots, b_5) = (0.1, 0, -0.85, 0, 0.1)$. Figure 2b shows that the function $S_{2,5}(x,\ y)$ has two locally minimal points in the invertible space Ω_2 (the gray triangle domain in Figure 2a, 2b).

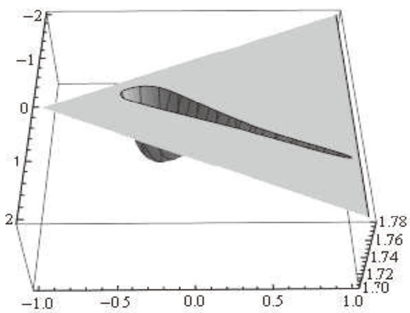

Figure 2a. Cross sections of $S_{2,5}(x,\ y)$ with $(b_1, \ldots, b_5) = (0.1, 0, -0.85, 0, 0.1)$.

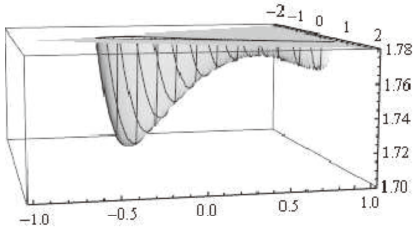

Figure 2b. Cross sections of $S_{2,5}(x,\ y)$ with $(b_1, \ldots, b_5) = (0.1, 0, -0.85, 0, 0.1)$.

In order to investigate the minimal point of the function $S_{2,q}(x,\ y)$, it is first necessary to consider its locally minimal points on the parameter space Ω_2. The locally minimal and maximal points satisfy the following equations,

$$\frac{\partial\ S_{2,q}(x,\ y)}{\partial_x} = 0, \tag{2.9}$$

$$\frac{\partial\ S_{2,q}(x,\ y)}{\partial_y} = 0. \tag{2.10}$$

In the next Section 3, we consider to solve these equations on x and y for each MA(q) process.

3. MA(2) model fittings to MA(q) processes.

3.1. MA(2) model fitting to the MA(0) process (white noise process).

In this case the estimated MA(2)-parameters are $(0, 0)$. For the function $S_{2,0}(x,\ y)$ is given by

$$S_{2,0}(x,\ y) = \frac{-1-y}{(-1+y)\ (1-x+y)\ (1+x+y)}. \tag{3.1}$$

and from (2.9) and (2.10) the locally minimal points of $S_{2,0}(x,\ y)$ in Ω_2 satisfy

$$x\ (1+y) = 0,$$

$$-x^2 + y\ (1+y)^2 = 0. \tag{3.2}$$

Hence it is seen that $x = 0$ and $y = 0$.

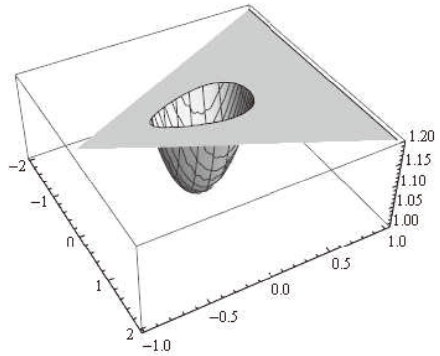

Figure 3.1. Cross section of $S_{2,0}(x, y)$ on Ω_2.

3.2. MA(2) model fitting to the MA(1) process.

The estimated MA(2) model has the MA-parameters $(b_1, 0)$.
For the function $S_{2,1}(x, y)$ is given by

$$S_{2,1}(x, y) = \frac{-1-y+2xb_1-b_1^2-yb_1^2}{(-1+y)(1-x+y)(1+x+y)}, \tag{3.3}$$

and from (2.9) and (2.10) the locally minimal points of $S_{2,1}(x, y)$ in Ω_2 should satisfy the equations

$$\begin{aligned} &(-1-y+xb_1)(x-b_1-yb_1)=0, \\ &-x^2+y+2y^2+y^3+xb_1+x^3b_1-2xyb_1- \\ &\qquad 3xy^2b_1-x^2b_1^2+yb_1^2+2y^2b_1^2+y^3b_1^2=0. \end{aligned} \tag{3.4}$$

Hence we can get $x = b_1$ and $y = 0$ by using the Mathematica (see Appendix 1).

3.3. MA(2) model fitting to the MA(2) process.

The estimated MA(2) model has the MA-parameters (b_1, b_2).
The function $S_{2,2}(x, y)$ is given by

$$\begin{aligned} S_{2,2}(x, y) = {} & \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \\ & \left(-1-y+2xb_1-b_1^2-yb_1^2-2x^2b_2+\right. \\ & \qquad \left. 2yb_2+2y^2b_2+2xb_1b_2-b_2^2-yb_2^2\right). \end{aligned} \tag{3.5}$$

From (2.9) and (2.10) the locally minimal points of $S_{2,2}(x, y)$ in Ω_2 should satisfy the equations

$$
\begin{aligned}
&-x - x\,y - x\,(1+y)\,b_1^2 - 2\,x\,(1+y)\,b_2 - x\,(1+y)\,b_2^2 \\
&\qquad + b_1\left(1 + x^2 + 2\,y + y^2 + \left(x^2 + (1+y)^2\right) b_2\right) = 0, \\
&-x^2 + y + 2\,y^2 + y^3 + \left(-x^2 + y\,(1+y)^2\right) b_1^2 - \left(x^4 - 2\,x^2\,y\,(2+y)\right. \\
&\qquad \left. + (1+y)^2\left(1+y^2\right)\right) b_2 + \left(-x^2 + y\,(1+y)^2\right) b_2^2 \\
&\qquad - b_1\left(-x\left(1 + x^2 - 2\,y - 3\,y^2\right) - x\left(1 + x^2 - 2\,y - 3\,y^2\right) b_2\right) = 0.
\end{aligned}
\tag{3.6}
$$

Hence we can get $x = b_1$ and $y = b_2$ in Ω_2 (see Appendix 2).

3.4. MA(2) model fitting to the invertible MA(3)(b_1, b_2, b_3) process with $b_3 \neq 0$.

The estimated MA(2) model has the MA-parameters $\left(\frac{-b_1 + b_2\,b_3}{-1 + b_3{}^2},\ \frac{-b_2 + b_1\,b_3}{-1 + b_3{}^2}\right)$.

For the function $S_{2,3}(x, y)$ is given by

$$
\begin{aligned}
S_{2,3}(x, y) = &\ \frac{1}{(-1+y)\,(1-x+y)\,(1+x+y)} \times \\
&\left(-1 - y + 2\,x\,b_1 - b_1^2 - y\,b_1^2 - 2\,x^2\,b_2 + 2\,y\,b_2 + 2\,y^2\,b_2 + \right. \\
&\ 2\,x\,b_1\,b_2 - b_2^2 - y\,b_2^2 + 2\,x^3\,b_3 - 4\,x\,y\,b_3 - 2\,x\,y^2\,b_3 - \\
&\left.\ 2\,x^2\,b_1\,b_3 + 2\,y\,b_1\,b_3 + 2\,y^2\,b_1\,b_3 + 2\,x\,b_2\,b_3 - b_3^2 - y\,b_3^2\right).
\end{aligned}
\tag{3.7}
$$

From (2.9) and (2.10) the locally minimal points of $S_{2,3}(x, y)$ in Ω_2 should satisfy the equations

$$
\begin{aligned}
&-x - x\,y - x\,(1+y)\,b_1^2 - x\,(1+y)\,b_2^2 + 3\,x^2\,b_3 - x^4\,b_3 - 2\,y\,b_3 + 4\,x^2\,y\,b_3 - \\
&\quad 5\,y^2\,b_3 + 2\,x^2\,y^2\,b_3 - 4\,y^3\,b_3 - y^4\,b_3 - x\,b_3^2 - x\,y\,b_3^2 + \\
&\quad b_1\left(1 + x^2 + 2\,y + y^2 + \left(x^2 + (1+y)^2\right) b_2 - 2\,x\,(1+y)\,b_3\right) + \\
&\qquad b_2\left(-2\,x\,(1+y) + \left(x^2 + (1+y)^2\right) b_3\right) = 0, \\
&-x^2 + y + 2\,y^2 + y^3 + \left(-x^2 + y\,(1+y)^2\right) b_1^2 + \left(-x^2 + y\,(1+y)^2\right) b_2^2 + \\
&\quad 2\,x\,b_3 - x^3\,b_3 + x^5\,b_3 + 2\,x\,y\,b_3 - 4\,x^3\,y\,b_3 + 3\,x\,y^2\,b_3 - \\
&\quad 2\,x^3\,y^2\,b_3 + 4\,x\,y^3\,b_3 + x\,y^4\,b_3 - x^2\,b_3^2 + y\,b_3^2 + \\
&\quad 2\,y^2\,b_3^2 + y^3\,b_3^2 - b_2\left(x^4 - 2\,x^2\,y\,(2+y) + (1+y)^2\left(1+y^2\right) - \right. \\
&\qquad \left. x\left(1 + x^2 - 2\,y - 3\,y^2\right) b_3\right) + b_1\left(-x\left(1 + x^2 - 2\,y - 3\,y^2\right) - \right. \\
&\qquad x\left(1 + x^2 - 2\,y - 3\,y^2\right) b_2 + \left(x^4 - 2\,x^2\,y\,(2+y) + \right. \\
&\qquad\qquad \left.\left. (1+y)^2\left(1+y^2\right)\right) b_3\right) = 0.
\end{aligned}
\tag{3.8}
$$

To solve these equations for x and y in Ω_2 is very difficult for us, we then use the Mathematica software. We first consider a special case when MA(3) model has parameters (0, 0, b_3) for $-1<b_3< 0$, $0 < b_3 < 1$.

From (3.8) we have the equations on x, y of MA(2) model parameters such that

$$\begin{aligned} f &= -x\,(1+y) - \left(x^4 + y\,(1+y)^2\,(2+y) - \right. \\ &\qquad \left. x^2\left(3+4y+2y^2\right)\right) b_3 - x\,(1+y)\,b_3^2 = 0, \\ g &= -x^2 + y + 2y^2 + y^3 + x\left(2 + x^4 + 2y + 3y^2 + \right. \\ &\qquad \left. 4y^3 + y^4 - x^2\left(1+4y+2y^2\right)\right) b_3 + \left(-x^2 + y\,(1+y)^2\right) b_3^2 = 0 \end{aligned} \tag{3.9}$$

The invertibility condition for the MA(3)($0, 0, b_3$) process is $1 - {b_3}^2 > 0$, and we can derive the solution $(0, 0)$ in Ω_2 (See Appendix 3.1).

When MA(3)(b_1, b_2, b_3) process with $b_3 \neq 0$, to solve the equations we first consider the polynomials of f and g, where f is the left hand side of the first equation of (3.9) and g is that of the second equation of (3.9). In order to get the common roots of f and g on x and y we use the resultant of the two polynomials f and g for each variables x and y. The resultant of f and g on y is given as

```
Ry = Resultant[f, g, x] // Factor
```

$$\begin{aligned} Ry = &-(1+y)^4\, b_3\, (-1+b_1-b_2+b_3)^4\, (1+b_1+b_2+b_3)^4 \\ &\left(-y+b_2-b_1 b_3+y b_3^2\right) \times \left(-y b_1^2 b_2^2 + b_2^3 + 2 y b_2^3 + y^2 b_2^3 + y^2 b_1^2 b_2^3 - \right. \\ &y b_2^4 - 2 y^2 b_2^4 - y^3 b_2^4 + 4 y b_1^3 b_3 - 4 b_1 b_2 b_3 - 10 y b_1 b_2 b_3 - \\ &4 y^2 b_1 b_2 b_3 - 4 y^2 b_1^3 b_2 b_3 + b_1 b_2^2 b_3 + 2 y b_1 b_2^2 b_3 + \\ &11 y^2 b_1 b_2^2 b_3 + 4 y^3 b_1 b_2^2 b_3 - y^2 b_1^3 b_2^2 b_3 + 4 y^2 b_1 b_2^3 b_3 + \\ &8 b_3^2 + 12 y b_3^2 + 6 y^2 b_3^2 + y^3 b_3^2 - 4 b_1^2 b_3^2 + 6 y b_1^2 b_3^2 - \\ &2 y^3 b_1^2 b_3^2 + y b_1^4 b_3^2 + 2 y^2 b_1^4 b_3^2 + y^3 b_1^4 b_3^2 + 4 b_2 b_3^2 - \\ &12 y b_2 b_3^2 - 15 y^2 b_2 b_3^2 - 4 y^3 b_2 b_3^2 - b_1^2 b_2 b_3^2 - \\ &2 y b_1^2 b_2 b_3^2 - 11 y^2 b_1^2 b_2 b_3^2 - 4 y^3 b_1^2 b_2 b_3^2 + \\ &4 b_2^2 b_3^2 - 6 y b_2^2 b_3^2 + 2 y^3 b_2^2 b_3^2 + y b_1^2 b_2^2 b_3^2 - \\ &4 y b_2^3 b_3^2 - 4 b_1 b_3^3 + 12 y b_1 b_3^3 + 15 y^2 b_1 b_3^3 + \\ &4 y^3 b_1 b_3^3 - b_1^3 b_3^3 - 2 y b_1^3 b_3^3 - y^2 b_1^3 b_3^3 + \\ &4 b_1 b_2 b_3^3 + 10 y b_1 b_2 b_3^3 + 4 y^2 b_1 b_2 b_3^3 - \\ &\left. 8 b_3^4 - 12 y b_3^4 - 6 y^2 b_3^4 - y^3 b_3^4\right) \end{aligned} \tag{3.10}$$

Also the resultant of f and g on x is given as

```
Rx = Resultant[f, g, y] // Factor
```

$$\begin{aligned} Rx = &\,x^4\, (-1+b_1-b_2+b_3)^4\, (1+b_1+b_2+b_3)^4 \left(-x+b_1-b_2 b_3+x b_3^2\right) \times \\ &\left(b_1 b_2^2 - x b_1^2 b_2^2 - x b_2^3 + 2 b_1 b_2^3 + 2 x^2 b_1 b_2^3 - \right. \\ &x b_1^2 b_2^3 - 2 x b_2^4 - x^3 b_2^4 + b_1 b_2^4 + x^2 b_1 b_2^4 - x b_2^5 - b_3 + x b_1 b_3 - \\ &2 b_1^2 b_3 + 2 x b_1^3 b_3 - b_1^4 b_3 + x b_1^5 b_3 - 3 b_2 b_3 - x^2 b_2 b_3 + 6 x b_1 b_2 b_3 - \\ &2 b_1^2 b_2 b_3 - 6 x^2 b_1^2 b_2 b_3 - b_1^4 b_2 b_3 - x^2 b_1^4 b_2 b_3 - 2 b_2^2 b_3 - 2 x^2 b_2^2 b_3 + \\ &5 x b_1 b_2^2 b_3 + 4 x^3 b_1 b_2^2 b_3 + x b_1^3 b_2^2 b_3 + b_2^3 b_3 + b_2^4 b_3 - 3 x b_3^2 + x^3 b_3^2 - \\ &2 b_1 b_3^2 + 8 x^2 b_1 b_3^2 + 4 x b_1^2 b_3^2 - 2 x^3 b_1^2 b_3^2 - b_1^3 b_3^2 + 2 x b_1^4 b_3^2 + x^3 b_1^4 b_3^2 - \\ &5 x b_2 b_3^2 - 4 x^3 b_2 b_3^2 - 5 x b_1^2 b_2 b_3^2 - 4 x^3 b_1^2 b_2 b_3^2 - 2 b_1^3 b_2 b_3^2 - \\ &2 x^2 b_1^3 b_2 b_3^2 - 4 x b_2^2 b_3^2 + 2 x^3 b_2^2 b_3^2 + 2 b_1 b_2^2 b_3^2 + 6 x^2 b_1 b_2^2 b_3^2 + \\ &x b_1^2 b_2^2 b_3^2 - 2 x b_2^3 b_3^2 + 5 x b_1 b_3^3 + 4 x^3 b_1 b_3^3 + 2 b_1^2 b_3^3 + \\ &2 x^2 b_1^2 b_3^3 + x b_1^3 b_3^3 + 2 b_2 b_3^3 - 8 x^2 b_2 b_3^3 - 6 x b_1 b_2 b_3^3 - b_1^2 b_2 b_3^3 + \\ &\left. 2 b_2^2 b_3^3 + 3 x b_3^4 - x^3 b_3^4 + 3 b_1 b_3^4 + x^2 b_1 b_3^4 - x b_2 b_3^4 + b_3^5\right) \end{aligned} \tag{3.11}$$

The conditions for getting the common roots of *f* and *g* on *y* and also on *x* are derived as

```
Reduce[-y + b2 - b1 b3 + y b3^2 == 0 , y, Reals]
```

$$(b_3 == -1 \ \&\& \ b_2 == -b_1) \ || \ (b_3 == 1 \ \&\& \ b_2 == b_1) \ || \ \left((b_3 < -1 \ || \ -1 < b_3 < 1 \ || \ b_3 > 1) \ \&\& \ y == \frac{-b_2 + b_1 b_3}{-1 + b_3^2}\right) \tag{3.12}$$

```
Reduce[-x + b1 - b2 b3 + x b3^2 == 0 , x, Reals]
```

$$(b_3 == -1 \ \&\& \ b_1 == -b_2) \ || \ (b_3 == 1 \ \&\& \ b_1 == b_2) \ || \ \left((b_3 < -1 \ || \ -1 < b_3 < 1 \ || \ b_3 > 1) \ \&\& \ x == \frac{-b_1 + b_2 b_3}{-1 + b_3^2}\right) \tag{3.13}$$

Therefore a solution on (x, y) of the equations (3.10) and (3.11) in the parameter space Ω_2 is given by

$$\left(\frac{-b_1 + b_2 b_3}{-1 + b_3^2}, \ \frac{-b_2 + b_1 b_3}{-1 + b_3^2} \right). \tag{3.14}$$

Here we should remark that the solution of the equations (3.10) and (3.11) is only one in the invertible space Ω_2. To prove this fact the polynomials *Ry1* in (3.10) and *Rx1* in (3.11) should have no roots in Ω_2, where

$$\begin{aligned} Ry1 = \ & -y b_1^2 b_2^2 + b_2^3 + 2 y b_2^3 + y^2 b_2^3 + y^2 b_1^2 b_2^3 - y b_2^4 - 2 y^2 b_2^4 - y^3 b_2^4 + \\ & 4 y b_1^3 b_3 - 4 b_1 b_2 b_3 - 10 y b_1 b_2 b_3 - 4 y^2 b_1 b_2 b_3 - 4 y^2 b_1^3 b_2 b_3 + \\ & b_1 b_2^2 b_3 + 2 y b_1 b_2^2 b_3 + 11 y^2 b_1 b_2^2 b_3 + 4 y^3 b_1 b_2^2 b_3 - y^2 b_1^3 b_2^2 b_3 + \\ & 4 y^2 b_1 b_2^3 b_3 + 8 b_3^2 \, 12 y b_3^2 + 6 y^2 b_3^2 + y^3 b_3^2 - 4 b_1^2 b_3^2 + \\ & 6 y b_1^2 b_3^2 - 2 y^3 b_1^2 b_3^2 + y b_1^4 b_3^2 + 2 y^2 b_1^4 b_3^2 + y^3 b_1^4 b_3^2 \, 4 b_2 b_3^2 - \\ & 12 y b_2 b_3^2 - 15 y^2 b_2 b_3^2 - 4 y^3 b_2 b_3^2 - b_1^2 b_2 b_3^2 - 2 y b_1^2 b_2 b_3^2 - \\ & 11 y^2 b_1^2 b_2 b_3^2 - 4 y^3 b_1^2 b_2 b_3^2 + 4 b_2^2 b_3^2 - 6 y b_2^2 b_3^2 + \\ & 2 y^3 b_2^2 b_3^2 + y b_1^2 b_2^2 b_3^2 - 4 y b_2^3 b_3^2 - 4 b_1 b_3^3 + \\ & 12 y b_1 b_3^3 + 15 y^2 b_1 b_3^3 + 4 y^3 b_1 b_3^3 - b_1^3 b_3^3 - 2 y b_1^3 b_3^3 - \\ & y^2 b_1^3 b_3^3 + 4 b_1 b_2 b_3^3 + 10 y b_1 b_2 b_3^3 + 4 y^2 b_1 b_2 b_3^3 - \\ & 8 b_3^4 - 12 y b_3^4 - 6 y^2 b_3^4 - y^3 b_3^4, \end{aligned} \tag{3.15}$$

$$\begin{aligned} Rx1 = \ & b_1 b_2^2 - x b_1^2 b_2^2 - x b_2^3 + 2 b_1 b_2^3 + 2 x^2 b_1 b_2^3 - x b_1^2 b_2^3 - 2 x b_2^4 - \\ & x^3 b_2^4 + b_1 b_2^4 + x^2 b_1 b_2^4 - x b_2^5 - b_3 + x b_1 b_3 - 2 b_1^2 b_3 + 2 x b_1^3 b_3 - \\ & b_1^4 b_3 + x b_1^5 b_3 - 3 b_2 b_3 - x^2 b_2 b_3 + 6 x b_1 b_2 b_3 - 2 b_1^2 b_2 b_3 - \\ & 6 x^2 b_1^2 b_2 b_3 - b_1^4 b_2 b_3 - x^2 b_1^4 b_2 b_3 - 2 b_2^2 b_3 - 2 x^2 b_2^2 b_3 + \\ & 5 x b_1 b_2^2 b_3 + 4 x^3 b_1 b_2^2 b_3 + x b_1^3 b_2^2 b_3 + b_2^3 b_3 + b_2^4 b_3 - 3 x b_3^2 + \\ & x^3 b_3^2 - 2 b_1 b_3^2 + 8 x^2 b_1 b_3^2 + 4 x b_1^2 b_3^2 - 2 x^3 b_1^2 b_3^2 - b_1^3 b_3^2 + \\ & 2 x b_1^4 b_3^2 + x^3 b_1^4 b_3^2 - 5 x b_2 b_3^2 - 4 x^3 b_2 b_3^2 - 5 x b_1^2 b_2 b_3^2 - \\ & 4 x^3 b_1^2 b_2 b_3^2 - 2 b_1^3 b_2 b_3^2 - 2 x^2 b_1^3 b_2 b_3^2 - 4 x b_2^2 b_3^2 + 2 x^3 b_2^2 b_3^2 + \\ & 2 b_1 b_2^2 b_3^2 + 6 x^2 b_1 b_2^2 b_3^2 + x b_1^2 b_2^2 b_3^2 - 2 x b_2^3 b_3^2 + 5 x b_1 b_3^3 + \\ & 4 x^3 b_1 b_3^3 + 2 b_1^2 b_3^3 + 2 x^2 b_1^2 b_3^3 + x b_1^3 b_3^3 + 2 b_2 b_3^3 - 8 x^2 b_2 b_3^3 - \\ & 6 x b_1 b_2 b_3^3 - b_1^2 b_2 b_3^3 + 2 b_2^2 b_3^3 + 3 x b_3^4 - x^3 b_3^4 + 3 b_1 b_3^4 + \\ & x^2 b_1 b_3^4 - x b_2 b_3^4 + b_3^5. \end{aligned} \tag{3.16}$$

They are both the 3rd polynomials on y and on x. It seems to be not easy for us that the equations $Ry1 = 0$ and $Rx1 = 0$ have no roots in Ω_2. Then we have not given the proof yet, and it must be a future work for us.

3.5. MA(2) model fitting to the MA(4) process.

Until now we have no counter example for uniqueness of the MA(2) parameters. From (2.6) the function $S_{2,4}(x, y)$ is given as

$$S_{2,4}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.17}$$

$$\big(-1 - y + 2 x b_1 - b_1^2 - y b_1^2 - 2 x^2 b_2 + 2 y b_2 + 2 y^2 b_2 + 2 x b_1 b_2 - b_2^2 - y b_2^2 + 2 x^3 b_3 - 4 x y b_3 - 2 x y^2 b_3 - 2 x^2 b_1 b_3 + 2 y b_1 b_3 + 2 y^2 b_1 b_3 + 2 x b_2 b_3 - b_3^2 - y b_3^2 - 2 x^4 b_4 + 6 x^2 y b_4 - 2 y^2 b_4 + 2 x^2 y^2 b_4 - 2 y^3 b_4 + 2 x^3 b_1 b_4 - 4 x y b_1 b_4 - 2 x y^2 b_1 b_4 - 2 x^2 b_2 b_4 + 2 y b_2 b_4 + 2 y^2 b_2 b_4 + 2 x b_3 b_4 - b_4^2 - y b_4^2\big).$$

A typical function $S_{2,4}(x, y)$ has the one local minimum point as follows. When MA(4) process has parameters {*0.1., 0., -0.85, -0.01*}, the function $S_{2,4}(x, y)$ is given as

$$S_{2,4}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.18}$$

$$\big(-1.7326 + 0.217 x + 0.17 x^2 - 1.702 x^3 + 0.02 x^4 - 1.9026 y + 3.404 x y - 0.06 x^2 y - 0.15 y^2 + 1.702 x y^2 - 0.02 x^2 y^2 + 0.02 y^3\big)$$

The cross section of the function $S_{2,4}(x, y)$ on Ω_2 is given in Figure 3.2, and the function has a locally minimal points in the MA(2)- parameter space Ω_2.

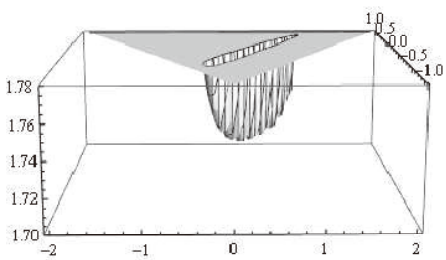

Figure 3.2. Cross section of $S_{2,4}(x, y)$ on Ω_2.

3.6. MA(2) model fitting to the MA(q) process for $q \geq 5$.

In this case we have a counter example for uniqueness of the MA(2) parameters. We consider the case $q = 5$, the MA(5) process. From (2.6) the function $S_{2,5}(x, y)$ is given as

$$S_{2,5}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.19}$$

$$\big(-1 - y + 2 x b_1 - b_1^2 - y b_1^2 - 2 x^2 b_2 + 2 y b_2 + 2 y^2 b_2 + 2 x b_1 b_2 - b_2^2 - y b_2^2 + 2 x^3 b_3 - 4 x y b_3 - 2 x y^2 b_3 - 2 x^2 b_1 b_3 + 2 y b_1 b_3 + 2 y^2 b_1 b_3 + 2 x b_2 b_3 - b_3^2 - y b_3^2 - 2 x^4 b_4 + 6 x^2 y b_4 - 2 y^2 b_4 + 2 x^2 y^2 b_4 - 2 y^3 b_4 + 2 x^3 b_1 b_4 - 4 x y b_1 b_4 - 2 x y^2 b_1 b_4 - 2 x^2 b_2 b_4 + 2 y b_2 b_4 + 2 y^2 b_2 b_4 + 2 x b_3 b_4 - b_4^2 - y b_4^2 + 2 x^5 b_5 - 8 x^3 y b_5 + 6 x y^2 b_5 - 2 x^3 y^2 b_5 + 4 x y^3 b_5 - 2 x^4 b_1 b_5 + 6 x^2 y b_1 b_5 - 2 y^2 b_1 b_5 + 2 x^2 y^2 b_1 b_5 - 2 y^3 b_1 b_5 + 2 x^3 b_2 b_5 - 4 x y b_2 b_5 - 2 x y^2 b_2 b_5 - 2 x^2 b_3 b_5 + 2 y b_3 b_5 + 2 y^2 b_3 b_5 + 2 x b_4 b_5 - b_5^2 - y b_5^2\big).$$

When MA(5) process has parameters {*0.1., 0., -0.85, -0.01., 0.1*}, the function $S_{2,5}(x, y)$ is derived as

$$S_{2,5}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.20}$$

$$\big(-1.7426 + 0.215 x + 0.34 x^2 - 1.702 x^3 - 3.46945 \times 10^{-18} x^4 + 0.2 x^5 - 2.0826 y + 3.404 x y + 1.38778 \times 10^{-17} x^2 y - 0.8 x^3 y - 0.34 y^2 + 2.302 x y^2 + 3.46945 \times 10^{-18} x^2 y^2 - 0.2 x^3 y^2 - 3.46945 \times 10^{-18} y^3 + 0.4 x y^3\big)$$

Figure 3.3 shows that the $S_{2,5}(x, y)$ has the two local minimum points in the MA(2) parameter space Ω_2, such that {$y=0.860, x=0.882$}, {$y= -0.276, x= -0.204$}. See Appendix 3.2.

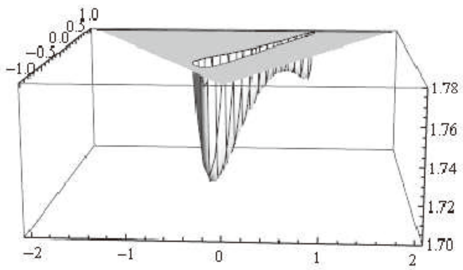

Figure 3.3. Cross section of $S_{2,5}(x, y)$ on Ω_2.

Also from (2.6) the function $S_{2,6}(x, y)$ is given as

$$S_{2,6}(x, y) = \frac{1}{(-1+y_2)(1-y_1+y_2)(1+y_1+y_2)} \times \tag{3.21}$$

$$\big(-1 - b_1^2 - b_2^2 - b_3^2 - b_4^2 - b_5^2 - b_6^2 + 2 b_1 y_1 + 2 b_1 b_2 y_1 + 2 b_2 b_3 y_1 +$$
$$2 b_3 b_4 y_1 + 2 b_4 b_5 y_1 + 2 b_5 b_6 y_1 - 2 b_2 y_1^2 - 2 b_1 b_3 y_1^2 - 2 b_2 b_4 y_1^2 -$$
$$2 b_3 b_5 y_1^2 - 2 b_4 b_6 y_1^2 + 2 b_3 y_1^3 + 2 b_1 b_4 y_1^3 + 2 b_2 b_5 y_1^3 + 2 b_3 b_6 y_1^3 -$$
$$2 b_4 y_1^4 - 2 b_1 b_5 y_1^4 - 2 b_2 b_6 y_1^4 + 2 b_5 y_1^5 + 2 b_1 b_6 y_1^5 - 2 b_6 y_1^6 - y_2 -$$
$$b_1^2 y_2 + 2 b_2 y_2 - b_2^2 y_2 + 2 b_1 b_3 y_2 - b_3^2 y_2 + 2 b_2 b_4 y_2 - b_4^2 y_2 +$$
$$2 b_3 b_5 y_2 - b_5^2 y_2 + 2 b_4 b_6 y_2 - b_6^2 y_2 - 4 b_3 y_1 y_2 - 4 b_1 b_4 y_1 y_2 -$$
$$4 b_2 b_5 y_1 y_2 - 4 b_3 b_6 y_1 y_2 + 6 b_4 y_1^2 y_2 + 6 b_1 b_5 y_1^2 y_2 + 6 b_2 b_6 y_1^2 y_2 -$$
$$8 b_5 y_1^3 y_2 - 8 b_1 b_6 y_1^3 y_2 + 10 b_6 y_1^4 y_2 + 2 b_2 y_2^2 + 2 b_1 b_3 y_2^2 - 2 b_4 y_2^2 +$$
$$2 b_2 b_4 y_2^2 - 2 b_1 b_5 y_2^2 + 2 b_3 b_5 y_2^2 - 2 b_2 b_6 y_2^2 + 2 b_4 b_6 y_2^2 - 2 b_3 y_1 y_2^2 -$$
$$2 b_1 b_4 y_1 y_2^2 + 6 b_5 y_1 y_2^2 - 2 b_2 b_5 y_1 y_2^2 + 6 b_1 b_6 y_1 y_2^2 - 2 b_3 b_6 y_1 y_2^2 +$$
$$2 b_4 y_1^2 y_2^2 + 2 b_1 b_5 y_1^2 y_2^2 - 12 b_6 y_1^2 y_2^2 + 2 b_2 b_6 y_1^2 y_2^2 - 2 b_5 y_1^3 y_2^2 -$$
$$2 b_1 b_6 y_1^3 y_2^2 + 2 b_6 y_1^4 y_2^2 - 2 b_4 y_2^3 - 2 b_1 b_5 y_2^3 + 2 b_6 y_2^3 - 2 b_2 b_6 y_2^3 +$$
$$4 b_5 y_1 y_2^3 + 4 b_1 b_6 y_1 y_2^3 - 6 b_6 y_1^2 y_2^3 + 2 b_6 y_2^4\big).$$

For an MA(6) process with the parameters $\{0.1., 0., -0.85, -0.01., 0.1, 0.1\}$ and with unit noise variance, we have

$$S_{2,6}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.22}$$

$$\big(-1.7526 + 0.235 x + 0.342 x^2 - 1.872 x^3 - 3.46945 \times 10^{-18} x^4 + 0.22 x^5 -$$
$$0.2 x^6 - 2.0946 y + 3.744 x y + 1.38778 \times 10^{-17} x^2 y - 0.88 x^3 y + 1. x^4 y -$$
$$0.342 y^2 + 2.532 x y^2 - 1.2 x^2 y^2 - 0.22 x^3 y^2 +$$
$$0.2 x^4 y^2 + 0.2 y^3 + 0.44 x y^3 - 0.6 x^2 y^3 + 0.2 y^4\big)$$

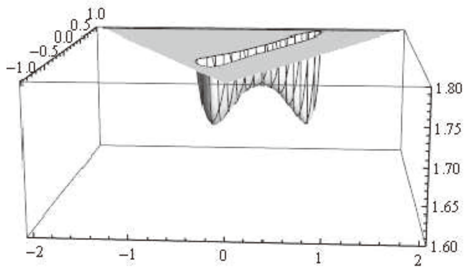

Figure 3.4. Cross section of $S_{2,6}(x, y)$ on Ω_2.

When $S_{2,5}\left(x, y \,\middle|\, \hat{b}_1, \ldots, \hat{b}_5\right)$ has 2 locally minimal points in the parameter space Ω_2, we want to know the condition of the MA(6) parameter b_6 that $S_{2,6}\left(x, y \,\middle|\, \hat{b}_1, \ldots, \hat{b}_5, b_6\right)$ also has 2 locally minimal points in Ω_2. Figure 3.3 and Figure 3.4 show that both $S_{2,5}\left(x, y \,\middle|\, \hat{b}_1, \ldots, \hat{b}_5\right)$ and $S_{2,6}\left(x, y \,\middle|\, \hat{b}_1, \ldots, \hat{b}_5, b_6\right)$ with $b_6 = 0.1$ have two locally minimal points in Ω_2. Then we focus on the difference between the two functions, and define a residual function such that

$$D_{2,6}(x,\ y \mid b_1,\ \cdots,\ b_5,\ b_6) = S_{2,6}(x,\ y \mid b_1,\ \cdots,\ b_5,\ b_6) - S_{2,5}(x,\ y \mid b_1,\ \cdots,\ b_5) \tag{3.23}$$

$$= \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times$$

$$b_6\left(2x\left(x^4 - x^2 y(4+y) + y^2(3+2y)\right)b_1 - 2\left(x^6 - y^3(1+y) + 3x^2y^2(2+y) - x^4 y(5+y) + \left(x^4 + y^2(1+y) - x^2 y(3+y)\right)b_2 + x\left(-x^2 + y(2+y)\right)b_3 + \left(x^2 - y(1+y)\right)b_4 - x b_5\right) - (1+y)b_6\right).$$

Figure 5 shows the cross section of the residual function $D_{2,6}\left(x,\ y \mid \hat{b_1},\ \cdots,\ \hat{b_5},\ 0.1\right)$ on Ω_2, where $\left\{\hat{b_1},\ \cdots,\ \hat{b_5}\right\} = \{0.1,\ 0,\ -0.85,\ -0.01,\ 0.1\}$.

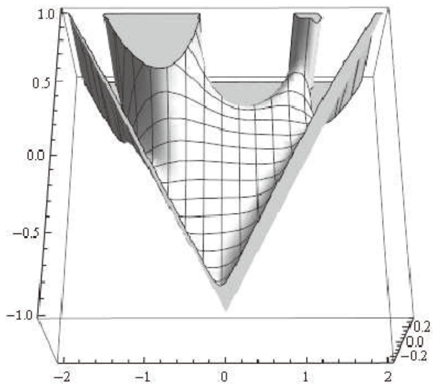

Figure 3.5. Cross section of $D_{2,6}(x,\ y)$ on Ω_2.

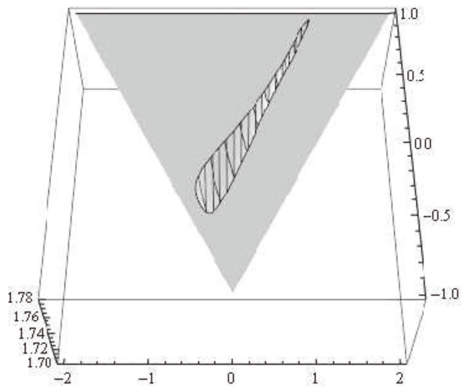

Figure 3.6. Cross section of $S_{2,5}(x,\ y)$ on Ω_2 (same function of Figure 3.3).

Figure 3.5 shows that the surface of $D_{2,6}(x,\ y)$ is relatively flat on the area of slash mark in Figure 3.6. Hence it is seen that the residual function $D_{2,6}(x,\ y)$ do not influence on the locally minimal points of $S_{2,5}(x,\ y)$. This imply that

$S_{2,6}(x,\ y)$ also have two locally minimal points on the invertible space, since $S_{2,5}(x,\ y) + D_{2,6}(x,\ y) = S_{2,6}(x,\ y)$. Similarly the function $S_{2,7}(x,\ y)$ will inherit the property from $S_{2,6}(x,\ y)$, and so on.

Therefore it may be conjectured that when MA(2) model is fitted to some data generated by MA(q) process for all $q \geq 5$, there are more than one MA(2) parameters estimated in the invertible space.

Next we consider the case when $S_{2,4}(x,\ y)$ in (3.20) and $S_{2,5}(x,\ y)$ in (3.22) such that $S_{2,4}(x,\ y)$ has only one locally minimal point but $S_{2,5}(x,\ y)$ does not have. In this case the residual function $D_{2,5}(x,\ y)$ is defined by

$$D_{2,5}(x,\ y \mid b_1,\ \ldots,\ b_5) = S_{2,5}(x,\ y \mid b_1,\ \ldots,\ b_5) - S_{2,4}(x,\ y \mid b_1,\ \ldots,\ b_4)\ . \tag{3.24}$$

$$= \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times$$

$$b_5 \left(-2 \left(x^4 + y^2 (1+y) - x^2 y (3+y)\right) b_1 + 2 \left(x^5 - x^3 y (4+y) + x y^2 (3+2y) + x \left(x^2 - y (2+y)\right) b_2 + \left(-x^2 + y + y^2\right) b_3 + x\, b_4\right) - (1+y) b_5\right).$$

When $b_1 = 0.1$, $b_2 = 0.0$, $b_3 = -0.85$, $b_4 = -0.01$ and $b_5 = 0.1$, we have $D_{2,5}(x,\ y \mid b_1,\ \ldots,\ b_5)$ and its graph, as follows.

$$D_{2,5}(x,\ y) = D_{2,5}(x,\ y \mid b_1,\ \ldots,\ b_5) \tag{3.25}$$

$$= \ = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times$$

$$\left(\left(0.1 \left(-0.1 (1+y) - 0.2 \left(x^4 + y^2 (1+y) - x^2 y (3+y)\right) + 2 \left(0. - 0.01 x + x^5 - x^3 y (4+y) + x y^2 (3+2y) - 0.85 \left(-x^2 + y + y^2\right)\right)\right)\right)\right).$$

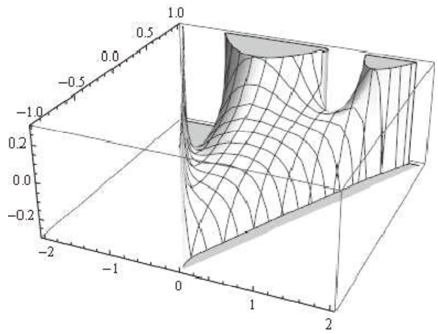

Figure 3.7. Cross section of $D_{2,5}(x,\ y)$ on Ω_2.

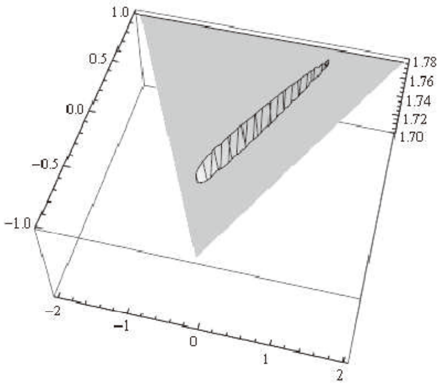

Figure 3.8. Cross section of $S_{2,4}(x,\ y)$ on Ω_2 (same function of Figure 3.2).

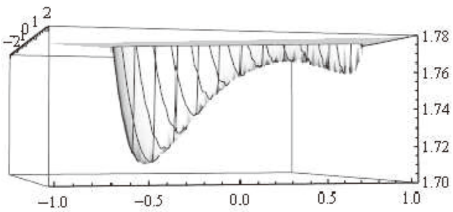

Figure 3.9. Cross section of $S_{2,5}(x,\ y)$ on Ω_2 (same function of Figure 3.6).

Since the graph of the shadow area in Figure 3.8 will be put on the back of the upper horse in Figure 3.7, we can say that the one local minimum point of $S_{2,4}(x,\ y)$ was divided into two. Under this situation, $S_{2,5}(x,\ y)$ has two local minimum points. The residual function $D_{2,5}(x,\ y)$ has grasped the key point for the number of the locally minimal points of $S_{2,5}(x,\ y)$.

Next we consider an MA(15) process with the parameters {0, -0.995, 0, (-0.995)^2, 0, (-0.995)^3, 0, (-0.995)^4, 0, (-0.995)^5, 0, (-0.995)^6, 0, (-0.995)^7, 0.01} and with unit noise variance. Mathematica shows that this MA(15) process is invertible, as follows.

```
In[ ]:= (*TimeSeriesInvertibility[
         MAProcess[{0, -0.995, 0, (-0.995)^2, 0, (-0.995)^3, 0, (-0.995)^4, 0.0,
           (-0.995)^5, 0, (-0.995)^6, 0.0, (-0.995)^7, 0.01}, 1]]*)

Out[ ]= True
```

In this case the function $S_{2,15}(x,\ y)$ is given as

$$S_{2,15}(x, y) = \frac{1}{(-1+y)(1-x+y)(1+x+y)} \times \tag{3.26}$$

$$-\big(\big(-9.16646 - 0.0193104\,x + 15.7439\,x^2 + 0.030866\,x^3 - 12.7111\,x^4 - 0.019505\,x^5 + 10.7849\,x^6 + 0.019603\,x^7 - 8.85908\,x^8 - 0.0197015\,x^9 + 6.93345\,x^{10} + 0.0198005\,x^{11} - 5.00799\,x^{12} - 0.0199\,x^{13} + 1.93104\,x^{14} + 0.02\,x^{15} - 24.9103\,y - 0.0617321\,x\,y + 38.1332\,x^2\,y + 0.0780199\,x^3\,y - 53.9247\,x^4\,y - 0.117618\,x^5\,y + 62.0135\,x^6\,y + 0.157612\,x^7\,y - 62.401\,x^8\,y - 0.198005\,x^9\,y + 55.0879\,x^{10}\,y + 0.2388\,x^{11}\,y - 25.1035\,x^{12}\,y - 0.28\,x^{13}\,y - 28.4549\,y^2 - 0.089381\,x\,y^2 + 77.4207\,x^2\,y^2 + 0.215535\,x^3\,y^2 - 143.671\,x^4\,y^2 - 0.433334\,x^5\,y^2 + 202.996\,x^6\,y^2 + 0.732519\,x^7\,y^2 - 232.293\,x^8\,y^2 - 1.1143\,x^9\,y^2 + 132.457\,x^{10}\,y^2 + 1.5799\,x^{11}\,y^2 - 1.93104\,x^{12}\,y^2 - 0.02\,x^{13}\,y^2 - 23.496\,y^3 - 0.117422\,x\,y^3 + 120.946\,x^2\,y^3 + 0.472442\,x^3\,y^3 - 286.966\,x^4\,y^3 - 1.22704\,x^5\,y^3 + 469.205\,x^6\,y^3 + 2.5464\,x^7\,y^3 - 363.694\,x^8\,y^3 - 4.599\,x^9\,y^3 + 21.2415\,x^{10}\,y^3 + 0.24\,x^{11}\,y^3 - 19.644\,y^4 - 0.157316\,x\,y^4 + 157.156\,x^2\,y^4 + 0.890032\,x^3\,y^4 - 454.561\,x^4\,y^4 - 2.92321\,x^5\,y^4 + 545.742\,x^6\,y^4 + 7.3164\,x^7\,y^4 - 86.8969\,x^8\,y^4 - 1.1\,x^9\,y^4 - 15.7925\,y^5 - 0.197609\,x\,y^5 + 174.502\,x^2\,y^5 + 1.51041\,x^3\,y^5 - 418.591\,x^4\,y^5 - 6.1544\,x^5\,y^5 + 162.207\,x^6\,y^5 + 2.4\,x^7\,y^5 - 11.9414\,y^6 - 0.238303\,x\,y^6 + 129.189\,x^2\,y^6 + 2.3765\,x^3\,y^6 - 135.173\,x^4\,y^6 - 2.52\,x^5\,y^6 - 6.93903\,y^7 - 0.2794\,x\,y^7 + 40.5519\,x^2\,y^7 + 1.12\,x^3\,y^7 - 1.93104\,y^8 - 0.14\,x\,y^8\big)\big).$$

The cross section of $S_{2,15}(x, y)$ on Ω_2 is in Figure 3.10 below, and the function has only one locally minimal point in Ω_2.

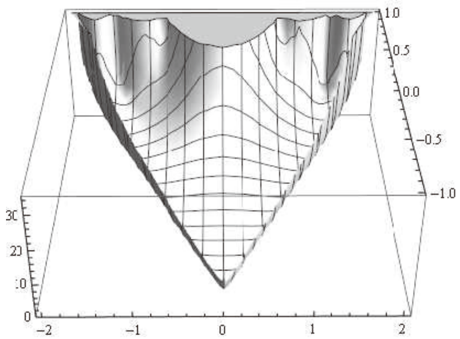

Figure 3.10. Cross section of $S_{2,15}(x, y)$ on Ω_2.

However, in the special case when $y = 0.995$, the function $S_{2,15}(x, 0.995)$ is given by

$$S_{2,15}(x, 0.995) = \tag{3.27}$$

$$\frac{(-1)}{(-1.92-x)(-1.92+x)} \times 12.5\,(-111.048 - 0.858341\,x + 539.087\,x^2 + 4.35185\,x^3 - 1090.87\,x^4 - 9.1284\,x^5 + 1102.89\,x^6 + 9.59068\,x^7 - 608.337\,x^8 - 5.51423\,x^9 + 186.266\,x^{10} + 1.76361\,x^{11} - 29.7377\,x^{12} - 0.294428\,x^{13} + 1.93104\,x^{14} + 0.02\,x^{15}).$$

From Figure 3.11 it is remarkable that $S_{2,15}(x, 0.995)$ has 6 locally minimal points in $-2 < x < 2$.

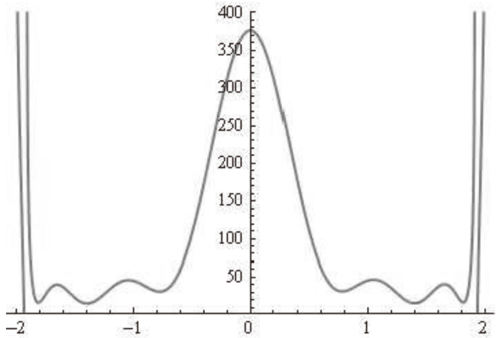

Figure 3.11. Graph of $S_{2,15}(x, 0.995)$ on $-2 < x < 2$.

4. Conclusions.

We have considered some problems for misspecified MA(2) model fittings to a data of Gaussian MA(q) process with unite noise variance. To estimate the MA(2)-parameters we introduce a function $S_{2,q}(x, y)$ for the conditional maximal likelihood estimation. Our discussions focus on the numbers of the locally minimal points of the function on the invertible space of MA(2) model. The considerations are as follows.

(1) If MA(2) model is fitted to MA(3) process with parameters (b_1, b_2, b_3), the MA(2)-parameters (x, y) are uniquely estimated in the invertible parameter space, or not. The answer is yes, and the estimator is given by $\left(\frac{-b_1+b_2 b_3}{-1+b_3^2}, \frac{-b_2+b_1 b_3}{-1+b_3^2}\right)$. However, though regrettable, a part of the proof is incomplete yet.

(2) If MA(2) model is fitted to MA(5) process and MA(6)process, then the MA(2)-parameters (x, y) are not uniquely estimated in the invertible parameter space. We showed the examples that there are two MA(2)-parameters in the invertible space. However, in the case of MA(4) process, the example of this kind is not yet found. Moreover, a proof of the estimated MA(2) model being restricted to one is not made, either.

(3) If MA(2) model is fitted to a data from MA(q) process for some $q \geq 5$ which the MA(2)-parameters are not uniquely estimated in the invertible space, then there is a data from MA($q+1$) process that the MA(2)-parameters are also not uniquely estimated in the invertible space, or not. In this paper we discussed the problem in the cases $q = 4$

and $q = 5$ by introducing the residual functions $D_{2,5}(x, y)$ and $D_{2,6}(x, y)$. Then it may be conjectured that when we consider the MA(2) model fitting to MA(q) process for $q \geq 5$, there exists a data of MA(q) process which has more than one estimators of MA(2)-parameters in the invertible parameter space.

(4) How many is the maximum of the number of the estimated MA(2) models fitted to MA(q) process? The maximum number is two despite former.

Appendixes.

Appendix 1. The estimated MA(2) model has the parameters (b_1, 0).

For the object function $S_{2,3}(x, y)$ we set it r0203.

In[ ]:= `(*r0201=r0203/.{b2→0,b3→0}*)`

Out[ ]= $$\frac{-1 - y + 2 x b_1 - b_1^2 - y b_1^2}{(-1 + y)(1 - x + y)(1 + x + y)}$$

In[ ]:=
```
(*dx00=∂xr0201//Simplify
 dy00=∂yr0201//Simplify*)
```

Out[ ]= $$\left(-2 x (1 + y) + 2 \left(x^2 + (1 + y)^2\right) b_1 - 2 x (1 + y) b_1^2\right) / \left((-1 + y)(1 - x + y)^2 (1 + x + y)^2\right)$$

Out[ ]= $$\left(2 \left(-x^2 + y (1 + y)^2 + x \left(1 + x^2 - 2 y - 3 y^2\right) b_1 + \left(-x^2 + y (1 + y)^2\right) b_1^2\right)\right) / \left((-1 + y)^2 (1 - x + y)^2 (1 + x + y)^2\right)$$

In[ ]:=
```
(*dnxy0={Numerator[dx00],Numerator[dy00]}//Factor
*)
```

Out[ ]= $$\{2 (-1 - y + x b_1)(x - b_1 - y b_1),\ 2 \left(-x^2 + y + 2 y^2 + y^3 + x b_1 + x^3 b_1 - 2 x y b_1 - 3 x y^2 b_1 - x^2 b_1^2 + y b_1^2 + 2 y^2 b_1^2 + y^3 b_1^2\right)\}$$

In[ ]:= `(*Solve[dnxy0=={0,0}&&1-y^2>0&&(x-x y) (-x+x y)+(1-y^2)^2>0&&1-b1^2>0, {x,y}]*)`

$$\{\{x \to \text{ConditionalExpression}[b_1, -1 < b_1 < 1],\ y \to \text{ConditionalExpression}[0, -1 < b_1 < 1]\}\}$$

Therefore we have the MA(2) parameters $x = b_1$ and $y = 0$.

Appendix 2. The estimated MA(2) model has the MA-parameters (b_1, b_2).

For the object function $S_{2,3}(x, y)$ we define vector d20 of its derivative on x and y, such that

```
d20 = {-x - x y - x (1 + y) b1^2 - x (1 + y) b2^2 + 3 x^2 b3 -  x^4 b3 - 2 y b3 + 4 x^2 y b3 - 5 y^2 b3 + 2 x^2 y^2 b3 -
    4 y^3 b3 - y^4 b3 - x b3^2 - x y b3^2 + b1 (1 + x^2 + 2 y + y^2 + (x^2 + (1 + y)^2) b2 - 2 x (1 + y) b3) +
    b2 (-2 x (1 + y) + (x^2 + (1 + y)^2) b3),
    -x^2 + y + 2 y^2 + y^3 + (-x^2 + y (1 + y)^2) b1^2 + (-x^2 + y (1 + y)^2) b2^2 +
    2 x b3 - x^3 b3 + x^5 b3 + 2 x y b3 - 4 x^3 y b3 + 3 x y^2 b3 - 2 x^3 y^2 b3 + 4 x y^3 b3 + x y^4 b3 -
    x^2 b3^2 + y b3^2 + 2 y^2 b3^2 + y^3 b3^2 - b2 (x^4 - 2 x^2 y (2 + y) + (1 + y)^2 (1 + y^2) - x (1 + x^2 - 2 y - 3 y^2) b3) -
    b1 (-x (1 + x^2 - 2 y - 3 y^2) - x (1 + x^2 - 2 y - 3 y^2) b2 + (x^4 - 2 x^2 y (2 + y) + (1 + y)^2 (1 + y^2)) b3)};
```

```
In[ ]:= (*d200=d20/.{b3→-0.0}
        *)
```

```
Out[ ]= {0. - x - x y - x (1 + y) b1^2 + (0. - 2 x (1 + y)) b2 -
          x (1 + y) b2^2 + b1 (1. + x^2 + 2 y + y^2 + (x^2 + (1 + y)^2) b2),
         0. - x^2 + y + 2 y^2 + y^3 + (-x^2 + y (1 + y)^2) b1^2 - (0. + x^4 - 2 x^2 y (2 + y) + (1 + y)^2 (1 + y^2)) b2 +
          (-x^2 + y (1 + y)^2) b2^2 - b1 (0. - x (1 + x^2 - 2 y - 3 y^2) - x (1 + x^2 - 2 y - 3 y^2) b2)}
```

```
In[ ]:= (*TimeSeriesInvertibility[MAProcess[{b1,b2},1]]*)
```

```
Out[ ]= 1 - b2^2 > 0 && (b1 - b1 b2) (-b1 + b1 b2) + (1 - b2^2)^2 > 0
```

```
In[ ]:= (*solv01 =
         Solve[d200 == {0, 0}&& 1-y^2>0&&(x-x y) (-x+x y)+(1-y^2)^2>0&&1-b2^2>0&&
           (b1-b1 b2) (-b1+b1 b2)+(1-b2^2)^2>0, {x, y}, Reals]*)
```

```
Out[ ]= {{x → ConditionalExpression[b1,
            (-2. < b1 < 0 && -1. - 1. b1 < b2 < 1.) || (0 < b1 < 2. && -1. + b1 < b2 < 1.)],
          y → ConditionalExpression[(0. + b1 b2)/b1, (-2. < b1 < 0 && -1. - 1. b1 < b2 < 1.) ||
            (0 < b1 < 2. && -1. + b1 < b2 < 1.)]}}
```

Therefore we have the MA(2) parameters $x = b_1$ and $y = b_2$.

Appendix 3.1.

```
In[ ]:= (*model03=MAProcess[{0,0,b3},1]
         {TimeSeriesInvertibility[model03],WeakStationarity[model03]}*)
```

```
Out[ ]= MAProcess[{0, 0, b3}, 1]
```

```
Out[ ]= {1 - b3^2 > 0 && (1 - b3^2)^2 > 0 && (1 - b3^2)^4 > 0, True}
```

```
(*d200=d20/.{b1→0,b2→0}//Simplify*)
```

$$\{-c x^4 - (1+c^2)\, x\,(1+y) - c\,y\,(1+y)^2\,(2+y) + c\,x^2\,(3+4y+2y^2),$$
$$-x^2 + y\,(1+y)^2 + c^2\,(-x^2 + y\,(1+y)^2) + c\,x\,(2+x^4+2y+3y^2+4y^3+y^4-x^2\,(1+4y+2y^2))\}$$

```
(*model0=MAProcess[{0,0,c},1]
 {TimeSeriesInvertibility[model0],WeakStationarity[model0]}*)
```

```
MAProcess[{0, 0, c}, 1]
```

$$\{1-c^2>0\ \&\&\ (1-c^2)^2>0\ \&\&\ (1-c^2)^4>0,\ \text{True}\}$$

```
(*solv0 = Reduce[d200 == {0, 0}&& 1-y^2>0&&(x-x y) (-x+x y)+(1-y^2)^2>0&&-1<c<1,
  {x, y}, Reals]*)
```

$$-1 < c < 1\ \&\&\ x == 0\ \&\&\ y == 0$$

We have the solution (0, 0) as the parameters of the MA(2) model in the invertible space.

Appendix 3.2.

We will solve the equations from the derivative of $S_{2,5}(x, y)$ on *x* and *y*. Here we set *r02065* = $S_{2,5}(x, y)$.

```
(*NSolve[D[r02065,{{x,y}}]=={0,0},{x,y},Reals]*)
```

```
{{y → 1.11251, x → 1.08466}, {y → 11.5387, x → -3.15815},
 {y → 0.860788, x → 0.882024}, {y → 8.72189, x → 2.90865}, {y → -0.682801, x → -0.717651},
 {y → 0.525101, x → 0.585409}, {y → -0.276583, x → -0.203849}}
```

Then there are 7 critical points, but in the invertible space there exist 3 points only.

```
(*sol=NSolve[D[r02065,{{x,y}}]=={0,0}&&1-y^2>0&&(x-x y) (-x+x y)+(1-y^2)^2>0,{x,y},Reals]*)
```

```
{{y → 0.860788, x → 0.882024}, {y → 0.525101, x → 0.585409}, {y → -0.276583, x → -0.203849}}
```

```
(*Sign@Det[D[r02065,{{x,y},2}]]/.sol*)
```

```
{1, -1, 1}
```

Hence we have two locally minimal points of $S_{2,5}(x, y)$ {y→0.860788, x→0.882024} and {y→-0.276583, x→-0.203849}.

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