On a Misspecified ARMA Model Fitting to a Data from Gaussian ARMA Process

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Abstract

This paper provides computer simulations concerning a misspecified ARMA(p, q) model fitting to (a data generated from) an ARMA(p, q+1) process, and also a misspecified ARMA(p+h, q) model fitting to an ARMA(p, q+k) process. They are mainly concerned with a problem for finding a number of locally maximal points of a conditional Gaussian likelihood function of the model when the sample size is large. It is detected that when p=q=1 and h=k=1, the conditional likelihood function of the ARMA(2,1) model has more than one locally maximal points if the model is fitted to a data generated from an ARMA(1,2) process. Also in the case where p=3, q=3 and k=3 it is seen that the conditional likelihood function of the ARMA(3,3) model has at least five locally maximal points if the model is fitted to a data generated from an ARMA(3,6) process.

Key words: ARMA(p,q) model fitting; conditional maximum likelihood function; Gaussian; locally minimal points; misspecification.

1. Introduction

In the previous papers ([10], [11]), we have considered the misspeciffied ARMA(1,1) model fitting and also MA(2) model fitting to a data generated from a stationary and invertible ARMA process. It is well known that when we fit an MA(1) model to some special time series data which does not follow MA(1) process, the MA(1) parameter does not always have an unique Gaussian quasi-maximum likelihood estimator. Tanaka and Huzii [13] obtained the conditions of AR(2) parameters on which the MA(1) quasi-likelihood function has more than one local maximal points in the invertible parameter space (-1, 1). Tanaka and Aoki [12] also gave the region for the AR(2) parameters on which the MA(1) conditional likelihood function has more than one local maximal points in the AR(2) parameter space shown in Figure 1. From Tanaka and Huzii [13], we have two local maximal points of the MA(1) conditional-likelihood function $F(x;a,b)=F(x)$, say, where x is an MA(1) parameter and a, b are AR(2) parameters. In order to have the conditions on which the function has two local maximal points in the parameter space, we should consider the differentiation $DF(x) = 0$, and we specified the case where the solution of the equation $DF(x) = 0$ changed from three to two. That is, the value of the resultant ([5]) was able to formalize the contour line for zero (the bifurcation set). We set the domain with a deep color surrounded with the curve of the shape of a wedge given in the upper part of Figure 1. Its boundary is the bifurcation set. It will be seen that the function $F(x;a,b)$ is locally a cusp (see Tanaka [10]).

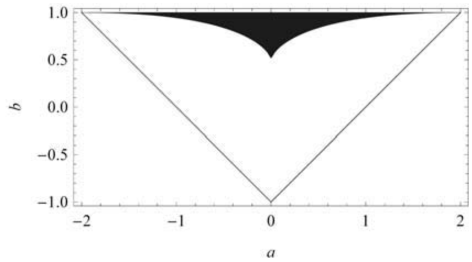

Figure 1. The domain for an MA(1) model fitting to an AR(2) process.

In the paper [10], we also considered a generalization of the MA(1) model fitting to an ARMA(1,1) model fitting. It related to incorrect identification of an ARMA(1,1) model and applied this to the time series which follows AR(2) process incorrectly. We have searched for the conditions of the coefficient parameter of AR(2) process in which two or more locally maximum points exist in quest of a conditional likelihood function paying attention to the number of the maximum points there. The graphs of the domain is shown in Figure 2.

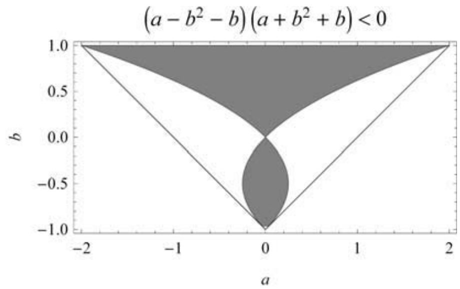

Figure 2. The domain for an ARMA(1,1) model fitting to an AR(2) process.

In the paper [11], we considered the ARMA(1,1) model fitting to an MA(2) process and study a problem similar to the ARMA(1,1) model fitting to AR(2) processes, and furthermore considered an MA(2) model fitting to AR(2) processes. Figure 3 show the domain of an MA(2) parameters where more than one locally maximum points of the conditional likelihood function of an ARMA(1,1) model exist.

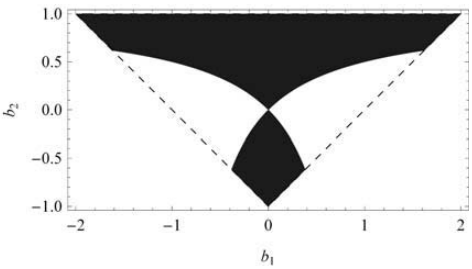

Figure 3. The domain for an ARMA(1,1) model fitting to an MA(2) process.

In this paper, we shall consider a higher order ARMA model fitting and try to know the answer of the six questions given in the next section. Since the theoretical arguments for these problems seems to be very difficult for us, so that we perform computer simulations to solve the problems. Our findings will lead to a conjecture that the conditional likelihood function of an ARMA(p+h, q) model has more than one locally maximum points in the stationary and invertible parameter space, if the model is fitted to a data generated from an ARMA(p, q+k) process for nonnegative integer p, h and for positive integer q, k.

2. ARMA model fittings to ARMA processes

In order to do computer simulations, we use the computer software, Time Series Pack for Mathmatica [7]. We then generate a scalar time series of length 40000 generated from the ARMA(p, q) process with parameters {AR{}, MA{}} by. Here the normally distributed noise has a mean 0 and a variance 1.0. It is well known that finding the exact maximum likelihood estimate of the ARMA model is generally very slow, so that we use a conditional likelihood. We fit a model to the data using the conditional maximum likelihood method with initial values of parameters the arguments of the model. By the way, if the incorrect-identified model is fitted to the data, it is known that the estimate may not be determined uniquely. That is, the estimated parameters of the model will be depend on the initial values for the likelihood method. This estimate must be one of the locally conditional maximum likelihood estimates. Therefore, in our simulations, we should try to find the locally conditional maximum likelihood estimate by changing initial values for the parameters. Here an interesting problem is how many misspecified models there are? Although the ARMA(1,1) model had been mainly considered until now, even when a true model was ARMA(2) and MA(2), the number of the estimated models using the conditional maximum likelihood method are at most two. It is imagined that the number of the models presumed will change by the model to fit and also by the true process to be fitted. Here we shall pay attention to ARMA(p, q) model fittings for small order p and q. Furthermore, we assume that the time series applied to these models follows the stationary and invertible Gaussian ARMA(p, q+1) process. Since the calculation is very complicated and generalities seems to be not easy for us, we shall consider a special case only.

Question 1: In the case when ARMA(1, 1) model fitting to an ARMA(1, 2) process, are there more than one models whose parameters locally maximize the conditional likelihood function of the ARMA(1, 1) model ?

Yes, there are. For example, we consider the ARMA(1,1) model fitting to the data from ARMA(1,2) process with parameters {(0.3), (0.0, 0.8)}. Then there are two ARMA(1,1) models such that ARMAModel[{-0.371686}, {0.919812}, 1.22836] and ARMAModel[{0.697173}, {-0.940481}, 1.48542] (the notation of the ARMAModel is given in Time Series Pack [7]). The spectra of the models are shown in Figure 4. The dotted line is the sample power spectrum of the data from the ARMA(1,2) process.

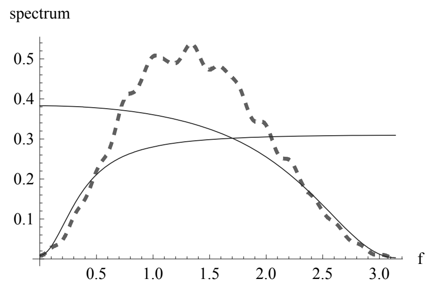

Figure 4. The spectra of the ARMA(1,1) models and the ARMA(1,2) process.

Question 2: In an ARMA(p, 1) model fitting to an ARMA(1, 2) process, are there more than one models whose parameters locally maximize the conditional likelihood of the ARMA(p, 1) model for nonnegative integer p?

To answer the question we set p = 19, for example, and we consider the ARMA(19, 1) model fitting to the ARMA(1, 2) = {(0.3), (0.0, -0.8)} process. Then two models are estimated such that ARMAModel[{1.2249, -1.07514, 0.976701, -0.856303, 0.776017, -0.674076, 0.609766, -0.521619, 0.461287, -0.398082, 0.350498, -0.294877, 0.251251, -0.19834, 0.164511, -0.123643, 0.091237, -0.0482513, 0.0237724}, {-0.929391}, 1.00597] and ARMAModel[{-0.553605, -0.545661, -0.441771, -0.43624, -0.351901, -0.343159, -0.273699, -0.261327, -0.213, -0.210049, -0.167429, -0.161462, -0.12847, -0.115596, -0.0851208, -0.0778697, -0.0528747, -0.0390475, -0.0112586}, {0.854079}, 1.00527]. Their theoretical spectra are displayed together in Figure 5. The dotted line is the sample power spectrum of a data from the original ARMA(1,2) process. In order to fit the ARMA(19, 1) model to a data using the conditional maximum likelihood method, we may set (recycle) the initial values for the parameters to the values of the estimated ARMA(18, 1) model, and we may set the new AR(19) parameter value 0, recurrently.

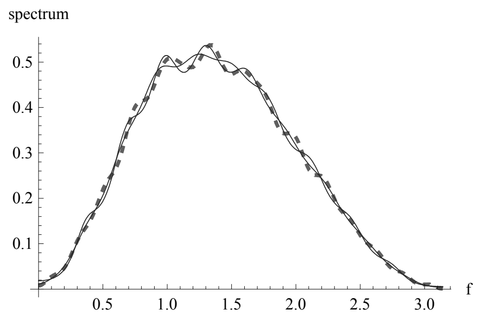

Figure 5. The spectra of the ARMA(19, 1) models and the ARMA(1, 2) process.

Therefore we may conjecture that for any p =1,2,3, .., in the ARMA(p, 1) model fitting to the ARMA(1, 2) process, we have at least two models whose parameters may locally maximize a conditional Gaussian likelihood function of the model.

Question 3: In an ARMA(2, 1) model fitting to an ARMA(2, 2) process, are there more than one models whose parameters locally maximize the conditional likelihood of the ARMA(2, 1) model ?

Yes, there are. For example, here we consider the ARMA(2,1) model fitting to ARMA(2,2) process with parameters {0.3, 0.1}, {0.0, -0.8}. Then there are two ARMA(2,1) models such that ARMAModel[{-0.44167, -0.230833}, {0.859173}, 1.11131] and ARMAModel[{0.840613, -0.393271}, {-0.773393}, 1.22183]. The spectra of the models are shown in Figure 6. The dotted line is the sample power spectrum of the data from the original ARMA(2,2) process.

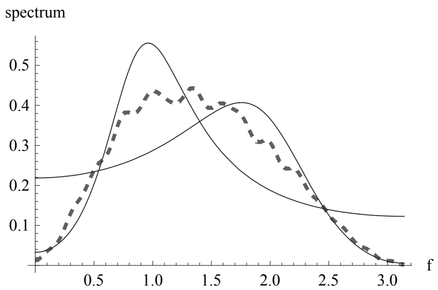

Figure 6. The spectra of the ARMA(2,1) models and the ARMA(2,2) process.

Question 4: In an ARMA(p, q) model fitting to ARMA(p, q+1) process, are there more than one models whose parameters locally maximize the conditional likelihood of the ARMA(p, q) model for any p= 0, 1, 2, ... and q= 1, 2, 3, ... ?

For example, we consider a case where p= q= 2; the ARMA(2,2) model fitting to an ARMA(2,3) process with parameters {{-0.8, -0.3}, {0.95, 0.7, 0.7}}. Then three models are estimated such that ARMAModel[{-1.3898, -0.582279}, {1.46333, 0.488967}, 1.30907], ARMAModel[{0.00854181, 0.631597}, {0.242396, -0.728351}, 1.36524] and ARMAModel[{0.0136486, -0.327005}, {-0.0892027, 0.795011}, 1.22078]. Their theoretical spectra are displayed together in Figure 7. The dotted line is the sample power spectrum of a data from the ARMA(2, 3) process.

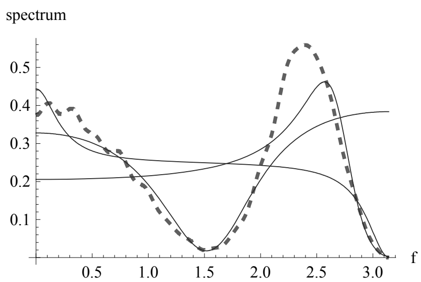

Figure 7. The spectra of the ARMA(2,2) models and the ARMA(2,3) process.

Also, we consider a case where p= 3 and q= 2; the AREMA(3,2) model fitting to an ARMA(3,3) process with parameters {AR{-0.8, -0.3, 0.2}, MA{0.1, -0.0, -0.8}}. Then two models are estimated such that ARMAModel[{-1.5428, -1.27859, -0.367502}, {1.05622, 0.876732}, 1.19391] and ARMAModel[{0.269621, 0.229349, -0.183879}, {-1.00264, 0.110153}, 1.06502]. Their theoretical spectra are displayed together in Figure 8. The dotted line is the sample power spectrum of a data from the ARMA(3,3) process.

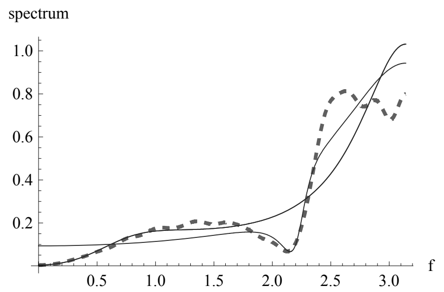

Figure 8. The spectra of the ARMA(3,2) models and the ARMA(3,3) process.

Next, we consider a case where p= 3 and q= 3; the ARMA(3,3) model fitting to an ARMA(3,4) process with parameters {{0.0, 0.5, 0.0}, {0.2, -0.1, -0.2, -0.8}}. The fitted ARMA(3,3) models are ARMAModel[{-0.889418, -0.282774, -0.14176}, {1.18918, 1.07928, 0.840654}, 1.10335] and ARMAModel[{0.905837, -0.291913, 0.141479}, {-0.801688, 0.680288, -0.835912}, 1.09631]. Their theoretical spectra are displayed together in Figure 9. The dotted line is the sample power spectrum of a data from the ARMA(3,4) process.

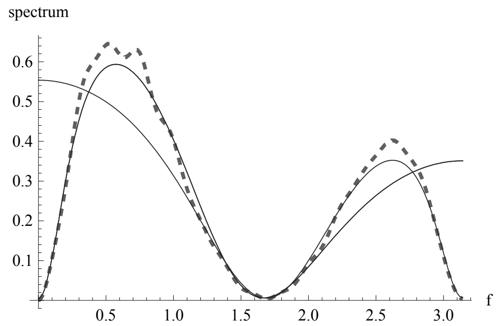

Figure 9. The spectra of the ARMA(3, 3) models and the ARMA(3, 4) process.

Also, we can obtained the similar results in a case when p= 3 and q= 4.

We next consider a case where p= 3 and q= 5; the ARMA(3, 5) model fitting to an ARMA(3, 6) process with parameters {{-0.1, -0.1, 0.2}, {-0.2, -0.1, -0.4, -0.3, 0.2, 0.5}}. Then there are four estimated ARMA(3,5) models such that ARMAModel[{0.606041, -0.383446, 0.288486}, {-0.956254, 0.466294, -0.506569, -0.172742, 0.521869}, 1.04585], ARMAModel[{-1.05134, -0.795684, -0.365877}, {0.873334, 0.539825, 0.0568548, -0.574721, -0.434987}, 1.12883], ARMAModel[{-0.734481, 0.563871, 0.568033}, {0.389982, -1.27943, -0.650362, 0.634194, 0.331622}, 1.28745], ARMAModel[{0.332183, 0.727747, -0.532979}, {-0.671249, -0.949919, 0.783654, 0.125677, -0.0224302}, 1.23976]. Their theoretical spectra are displayed together in Figure 10. The dotted line is also the sample power spectrum of a data from the ARMA(3, 6) process.

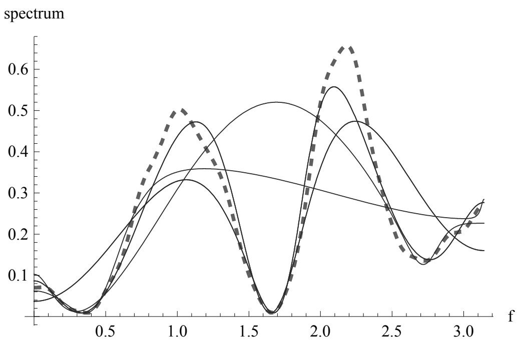

Figure 10. The spectra of the ARMA(3, 5) models and the ARMA(3, 6) process.

We have considered the question 4 in the case when p=3 only. But the other AR order (p >3) may also give the similar results.

Question 5: In an ARMA(p+h, q) model fitting to ARMA(p, q+1) process for fixed the orders p= 0, 1, 2, ... and q= 1, 2, 3, ... , are there more than one models whose parameters locally maximize the conditional likelihood of the ARMA(p+h, q) model for each h= 0, 1, 2 ,... ?

Here we consider a case where p=3, q=5 and h=1 ; an ARMA (4, 5) model fitting to an ARMA (3, 6) process such that ARMAModel[{-0.14, -0.02, 0.26}, {-0.14, -0.17, -0.12, 0.33, 0.29, 0.78}, 1.0]. Then there are four estimated ARMA(4,5) models such that ARMAModel[{0.876811, -0.666947, 0.637916, -0.302816}, {-1.03081, 0.818281, -0.604773, 0.246149, 0.312091}, 1.51736], ARMAModel[{-1.48217, -1.44027, -0.754441, -0.234866}, {1.58179, 1.60634, 1.00088, 0.388645, -0.147829}, 1.55906], ARMAModel[{0.118066, 0.649089, 0.169122, -0.399842}, {-0.156198, -0.85155, 0.0326855, 0.825754, -0.105601}, 1.48834] and ARMAModel[{-1.05973, 0.166968, 0.880341, 0.443437}, {1.06921, -0.267944, -0.814741, -0.0139132, 0.168352}, 1.56212]. Their theoretical spectra are displayed together in Figure 11. The dotted line is also the sample power spectrum of a data from the ARMA(3, 6) process.

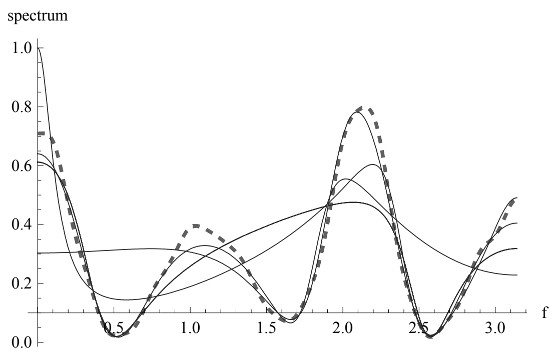

Figure 11. The spectra of the ARMA(4, 5) models and the ARMA(3, 6) process with parameters {{-0.14, -0.02, 0.26}, {-0.14, -0.17, -0.12, 0.33, 0.29, 0.78}}.

Question 6: How many misspecified models are there at most whose parameters locally maximize the conditional likelihood, when an ARMA(p+h, q) model is fitting to the data from an ARMA(p, q+k) process for h=0, 1, 2, ... and k = 1, 2, 3, ... ?

We have discovered a misspecified model fitting which has at least 5 sets of ARMA parameters. Here we consider only a case where p=3, q =3, h=0 and k=3; an ARMA(3, 3) model fitting to the data from an ARMA(3, 6) process with parameters {{-0.2, -0.1, 0.2}, {-0.0, -0.0, -0.0, 0.4, 0.1, 0.5}}. Then at least five ARMA(3, 3) models may be estimated by using the conditional maximum likelihood method, such that ARMAModel[{1.77951, -1.18046, 0.295838}, {-2.08279, 1.72307, -0.463262}, 1.16073], ARMAModel[{0.325232, 0.720903, -0.484079}, {-0.655757, -0.564475, 0.886864}, 1.16203], ARMAModel[{0.218219, -0.0932381, 0.630566}, {-0.315985, 0.13663, -0.381219}, 1.31657], ARMAModel[{0.973428, -0.164411, -0.0928348}, {-1.26674, 0.420188, 0.303678}, 1.15949] and ARMAModel[{-1.84197, -1.51373, -0.518573}, {1.8033, 1.32417, 0.284719}, 1.29156]. Also their theoretical spectra of the ARMA(3, 3) models are displayed together in Figure 13. The dotted line is the sample power spectrum of the data generated from the ARMA(3, 6) process.

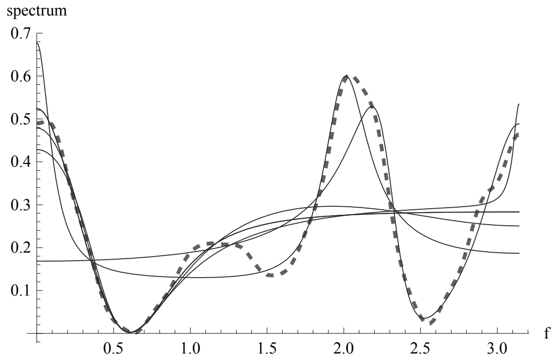

Figure 12. The spectra of the ARMA(3,3) models and the ARMA(3, 6) process with parameters {{-0.2, -0.1, 0.2}, {0.0, 0.0, 0.0, 0.4, 0.1, 0.5}}.

The simulation provides us that if we consider the case where increasing k (=1,2,3...) in the ARMA(p+h, q) model fitting to an ARMA(p, q+k) process for given p, q and h, the estimated ARMA(p+h, q) models show a tendency to grow in number.

4. Conclusions

In this paper, we have considered the six problems (Questions 1~ 6) for misspecified ARMA model fittings to the data from stationary and invertible Gaussian ARMA processes following to the previous paper [11] in 2013. The theoretical arguments for these problems seems to be very difficult for us, so that we have done computer simulations to solve the problems.

For Question 1, we obtained a positive result. We found two ARMA(1,1) models whose parameters locally maximize the conditional Gaussian likelihood for the data generated by ARMA(1,2) process. But, we did not consider the problem theoretically for finding the conditions of ARMA(1,2) parameters on which two or more locally maximum points exist in quest of the conditional likelihood function paying attention to the number of the maximum points.

For Question 2, we considered ARMA(p,1) model fittings to the data from an ARMA(1,2) process for integer p=1,2,...,19. Our findings will lead to the conjecture that for any positive integer p, there are at least two ARMA(p, 1) models estimated by using the conditional maximum likelihood method. We can confirm the positive answer for the cases where p= 0,1,2,...,19.

For Question 3, we considered ARMA(2,1) model fittings to the data from an ARMA(2,2) process and we obtained a positive result. There are at least two ARMA(2,1) models whose parameters locally maximize the conditional Gaussian likelihood for the data generated by an ARMA(2,2) process.

For Question 4, we considered the simulations for ARMA(p,q) model fittings to the data from an ARMA(p, q+1) process in the cases where p= 2,3 and q= 2,3,4. They will lead to the conjecture that for any positive integer p and q, there are at least two ARMA(p,q) models estimated using the conditional maximum likelihood method.

For Question 5, we considered the simulation study for an ARMA(p+h, q) model fitting to the data from the ARMA(p, q+1) process in the case where p= 3, q= 5 and h= 1. It will lead to the conjecture that for any positive integer p, q and h= 0,1,2.., there are at least two ARMA(p+h, q) models which are fitted to a data generated from an ARMA(p, q+1) process by using the conditional maximum likelihood method.

For Question 6, we performed simulation studies for ARMA(p+h, q) model fittings to the ARMA(p+k, q+1) process in the cases where p= 3, q= 3, h= 0 and k= 3. Our simulation studies will lead to the conjecture that for any positive integer p, q and h, k= 0,1,2,.., there are at least two ARMA(p+h, q) models which fit to the data from an ARMA(p+k, q+1) process by using the conditional maximum likelihood method. In the simulation studies, we can find an example in which (at least) five ARMA(3,3) models are estimated using the conditional maximum likelihood method by changing the initial values appropriately. We want to know that five is the maximum number for the misspecified models whose parameters are the locally maximum conditional likelihood estimators. Moreover, we have discovered an example in which six models are estimated for an ARMA(3,3) model fitting to a data from an ARMA(3,6) process. But the process was not invertible. Any theoretical consideration was not developed from this result. It must be a future subject for us.

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