On the generalization of von Mises distributions on the circle

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Abstract. We shall discuss a problem of how to construct angular distributions on the circle considering the generalization of the well-known von Mises distributions. We introduce three methods of the transformations to polar co-ordinates from a bivariate normal distribution instead of the radial projection of the distribution. Furthermore, the scale mixture of the obtained distributions with the inverse Gamma distribution for the mixing distribution is considered.

Keywords: von Mises distributions, circular data, Wrapped normal distribution, Wrapped Cauchy distribution, scale mixture of distributions

1. Introduction

It is well-known that analogous to the normal distributions on the line, the von Mises distributions are important families of continuous distributions on the circle and they play a key role in statistical inference on the circle. Thus it is called the circular normal distribution of angles.

In this papaer, we shall discuss a problem of how to construct angular distributions considering the generalization of the von Mises distributions on the circle. In section 3, to construct angular distributions we introduce three methods of the transformations to polar co-ordinates from a bivariate normal distribution instead of the radial projection of the distribution. Furthermore, in section 4, we consider the scale mixture of the obtained distributions with the inverse Gamma distribution for the mixing distribution.

A distribution on the line has a corresponded circular distribution, and thus it seems to be possible to derive a new distribution on the line from a new angular distribution by using the inverse transformation. In section 5, we shall try to derive a distribution on the line from a new angular distribution obtained in section 4.

Here axial data (observations of axes such that the angle θ and $\theta + \pi$ are equivalent) are considered to be circular data by transforming the angle θ to 2θ.

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2. Preliminaries and Notation

The von Mises distribution $\mathrm{M}(\mu, \kappa)$ has a probability density function, see Fisher[1] also Mardia and Jupp[3],

$$\mathrm{g}(\theta; \mu, \kappa) = \frac{e^{-\kappa \operatorname{Cos}[\theta-\mu]}}{2\pi \operatorname{BesselI}[0, \kappa]}, \quad \text{for } -\pi \le \theta \le \pi,$$

where $-\infty < \mu < \infty$, $0 < \kappa$, and BesselI[0, κ] denotes the modified Bessel function of the first kind and order 0, which is defined by,

$$\text{BesselI}[0, \kappa] = \frac{1}{2\pi}\int_0^{2\pi} e^{-\kappa \text{Cos}[\theta]}\, d\theta,$$

see Gradshteyn and Ryzhik[2]. It has power series expansion

$$\text{BesselI}[0, \kappa] = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{\kappa}{2}\right)^{2n},$$

see Titchmarsh[7].

The parameter μ is the mean direction and the parameter κ is known as the concentration parameter. Examples of the shape of the distribution are given in Figure 2.1. The distribution is unimodal and is symmetrical about $\theta = \mu$.

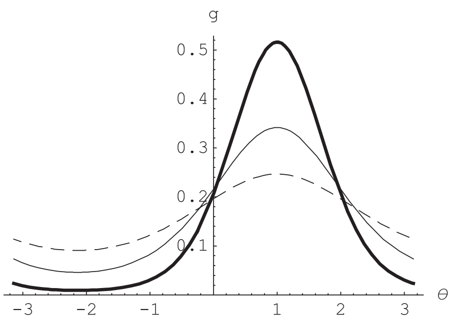

Figure 2.1: Density of the von Mises distribution M(0, κ) for $\mu = 1$,
and $\kappa = 0.5, 1, 2$ (bold, plain, dashed).

It is seen that when $\kappa = 0$, the von Mises distribution M(μ, κ) is the uniform distribution, and that M(μ, κ) becomes concentrated at the point $\theta = \mu$ as $\kappa \to \infty$. Also if a random variable Θ belongs to the von Mises distribution M(μ, κ) , the normalized variable $(\Theta - \mu) / \sqrt{\kappa}$ can be approximated by the standard normal distribution N(0, 1) as $k \to \infty$.

For small κ, the von Mises distribution M(μ, κ) is approximated by the cardioid distribution with the same mean μ direction.

(1) Cardioid distribution C(μ, ρ) :

$$f(\theta; \mu, \rho) = \frac{\{1 + 2\rho\, \text{Cos}(\theta - \mu)\}}{2\pi}, \quad -1/2 \le \rho \le 1/2.$$

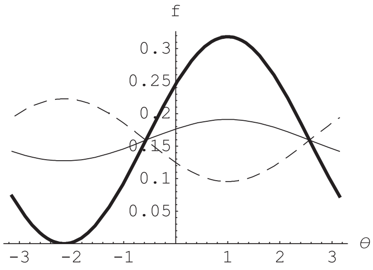

Figure 2.2: Density of the Cardioid distribution $C(\mu, \rho)$ for $\mu = 1$,
and ρ = -1/5, 1/10, 1/2 (bold, plain, dashed).

Any von Mises distribution can be approximated by a wrapped normal distribution for intermediate values of κ.

(2) Wrapped normal distribution $WN(\mu, \rho)$ is obtained by wrapping the $N(\mu, \sigma^2)$ distribution onto the circle, where $\sigma^2 = -2 \log \rho$. The probability density is given by

$$\phi_W(\theta; \mu, \rho) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp\left\{\frac{-(\theta-\mu+2\pi n)^2}{2\sigma^2}\right\}$$

$$= \frac{1}{2\pi}\left\{1 + 2\sum_{n=1}^{\infty} \rho^{n^2} \operatorname{Cos} n(\theta-\mu)\right\},$$

where $0 \le \rho \le 1$.

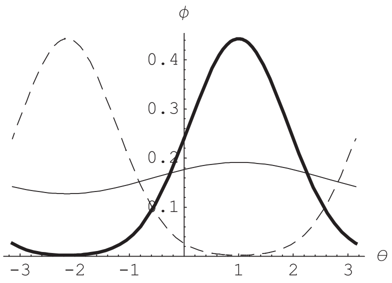

Figure 2.3: Density of the Wrapped normal distribution $WN(\mu, \rho)$ for $\mu = 1$,
and ρ = -2/3, 1/10, 2/3 (bold, plain, dashed).

Also the von Mises distribution is close to the wrapped normal distribution with the same mean direction μ.

(3) Wrapped Cauchy distribution $WC(\mu, \rho)$ is obtained by wrapping the Cauchy distribution onto the circle, where $\rho = e^{-a}$. The probability density is given by

$$c(\theta;\mu,\rho) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\left\{\frac{a}{a^2+(\theta-\mu+2\pi n)^2}\right\}$$

$$= \frac{1}{2\pi}\left\{1+2\sum_{n=1}^{\infty}\rho^n \operatorname{Cos} n(\theta-\mu)\right\}$$

$$= \frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\operatorname{Cos}(\theta-\mu)}.$$

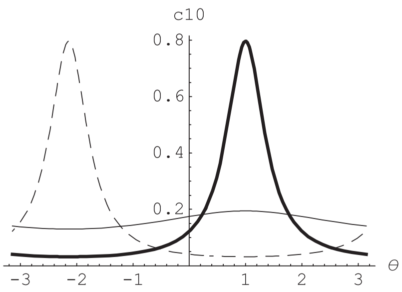

Figure 2.4: Density of the Wrapped Cauchy distribution WC(μ, ρ) for μ = 1, and ρ = -2/3, 1/10, 2/3 (bold, plain, dashed).

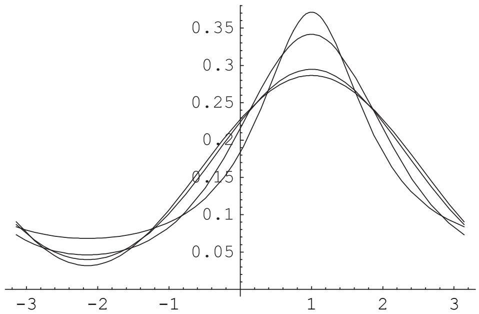

Figure 2.5: Density functions of the von Mises distribution, the Cardioid distribution, the Wrapped normal distribution and the Wrapped Cauchy distribution WC(μ, ρ) with μ = 1 and ρ = 1/3.

3. Derivation of the distributions on the circle from the bivariate Normal distribution using the transformation to Polar co-ordinates

Usually distributions on the circle can be obtained by radial projection of distributions on the plane. An important instance is the case of a bivariate normal distribution, and then the projected normal distribution (or angular Gaussian) distribution is obtained.

The probability density function of the angular central Gaussian distribution is

$$p(\theta;\theta,\delta) = \frac{\sqrt{1-b^2}}{2\pi\{1-b\,\mathrm{Cos}\,2(\theta-\delta)\}}, \quad -\pi \le \theta < \pi,$$

where $-1 < b < 1$ and $-\pi \le \delta < \pi$,
see Mardia and Jupp[4] and also Johnson, Kotz and Balakrishnan[3].

We here consider different ways for obtaining the distributions on the circle, and we begin by setting up the appropriate bivariate Normal distribution:

$$\phi(x_1, x_2; \mu_1, \mu_2, \sigma, \rho)$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} *$$

$$\exp\left[-\frac{1}{2(1-\rho^2)\sigma^2} * (x_1^2 + x_2^2 + \mu_1^2 + 2x_2(\rho\mu_1 - \mu_2) - 2\rho\mu_1\mu_2 + \mu_2^2 - 2x_1(\rho x_2 + \mu_1 - \rho\mu_2))\right],$$

for $-\infty < x_1 < \infty$, and $-\infty < x_2 < \infty$,
where $-1 < \rho < 1$, $\sigma > 0$, $-\infty < \mu_1 < \infty$, and $-\infty < \mu_2 < \infty$,
see Johnson, Kotz and Balakrishnan [3].

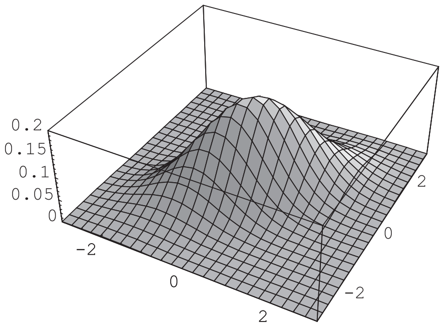

Figure 3.0: The density function of $\phi(x_1, x_2; \mu_1, \mu_2, \sigma, \rho)$ with the parameters $\{\sigma\to 1, \rho\to -0.4, \mu_1\to -\pi/6, \mu_2\to \pi/6\}$.

[1] We first consider the well-known transformation to polar co-ordinates;

$$\{x_1 \to r\,\mathrm{Cos}[\theta],\ x_2 \to r\,\mathrm{Sin}[\theta]\}.$$

It is known that $R = \sqrt{X_1^2 + X_2^2}$ represents the distance of (X_1, X_2) from the origin, while $\Theta = \arctan(X_2 / X_1)$ represents the angle of (X_1, X_2) with respect to the X_1 axis. Then $R = r > 0$ and $\Theta = \theta \in \{\theta : -\pi < \theta < \pi\}$. The joint distribution of R and Θ is given by the transformation method and the desired joint density is

$$\mathrm{g1}\,(\mathrm{r}, \theta\,; \lambda, \mu, \sigma, \rho)$$

$$= \frac{\mathrm{r}}{2\,\pi\,\sqrt{1-\rho^2}\;\sigma^2} \;*$$

$$\exp\left(-\frac{\mathrm{r}^2+\lambda^2-2\,\mathrm{r}\,\lambda\,\mathrm{Cos}[\theta-\mu]-\mathrm{r}^2\,\rho\,\mathrm{Sin}[2\,\theta]+\lambda\,\rho\,(-\lambda\,\mathrm{Sin}[2\,\mu]+2\,\mathrm{r}\,\mathrm{Sin}[\theta+\mu])}{2\,(1-\rho^2)\,\sigma^2}\right),$$

where $0 < \mathrm{r} < \infty$, $-\pi < \theta < \pi$, $\sigma > 0$, $\lambda > 0$, $-1 < \rho < 1$ and $-\pi \leq \mu < \pi$.

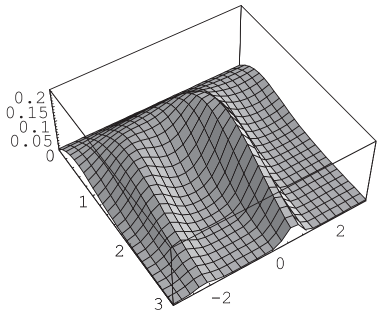

Figure 3.1: The density function of $\mathrm{g21}(\mathrm{r}, \theta\,; \lambda, \mu, \sigma, \rho)$
for $\{\sigma\to1, \rho\to0.6, \lambda\to1.0, \mu\to0\}$.

Then the conditional density function for the variable $\Theta = \theta$ given $\mathrm{R} = \mathrm{r}$ can be given by

$$\mathrm{gc1}\,(\theta\,; \lambda, \mu, \sigma, \rho, \mathrm{r}) =$$

$$\frac{1}{\mathrm{J}}\;\exp\left(-\frac{\mathrm{r}^2+\lambda^2-2\,\mathrm{r}\,\lambda\,\mathrm{Cos}[\theta-\mu]-\mathrm{r}^2\,\rho\,\mathrm{Sin}[2\,\theta]+\lambda\,\rho\,(-\lambda\,\mathrm{Sin}[2\,\mu]+2\,\mathrm{r}\,\mathrm{Sin}[\theta+\mu])}{2\,(1-\rho^2)\,\sigma^2}\right),$$

where

$$\mathrm{J} = \int_{-\pi}^{\pi} e^{\frac{\mathrm{r}^2+\lambda^2-2\,\mathrm{r}\,\lambda\,\mathrm{Cos}[\theta-\mu]-\mathrm{r}^2\,\rho\,\mathrm{Sin}[2\,\theta]+\lambda\,\rho\,(-\lambda\,\mathrm{Sin}[2\,\mu]+2\,\mathrm{r}\,\mathrm{Sin}[\theta+\mu])}{2\,(-1+\rho^2)\,\sigma^2}}\,d\theta$$

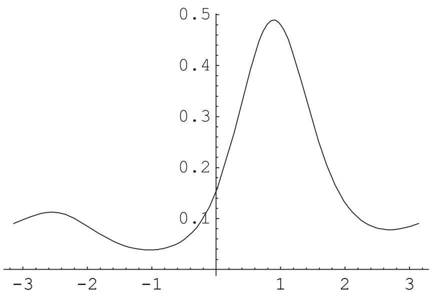

Figure 3.2: The density function of gc1(θ ; r, σ, ρ) for
$\{r\to 2, \sigma\to 1, \rho\to 0.3, \lambda\to 1/2, \mu\to\pi/3\}$.

The special case when $\lambda\to 0$, we have

$$\mathrm{gc10}(\theta;\sigma,\rho,\mathrm{r}) = \frac{e^{-\frac{r^2 \rho \,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]}{(-1+\rho^2)\sigma^2}}}{2\pi\,\mathrm{BesselI}\left[0, \frac{r^2\rho}{2(-1+\rho^2)\sigma^2}\right]},$$

which is the axial version of the von Mises distribution M(0, κ) on the half circle ($-\pi/2 \le \theta \le \pi/2$) when we set $\kappa = \frac{r^2\rho}{2(-1+\rho^2)\sigma^2} > 0$.

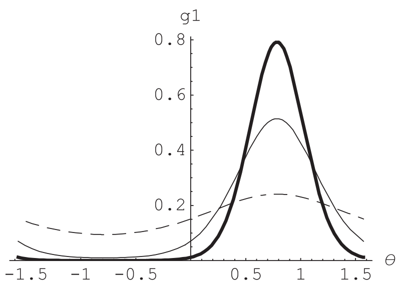

Figure 3.3: The density function of gc10(θ ; r, σ, ρ)
for $\sigma\to 1, \rho\to 0.6$, r$\to$\{1, 2, 3\} (bold, plain, dashed).

Therefore the distribution with the density gc1(θ ; λ, μ, σ, ρ, r) can be considered as a generalization of the von Mises distribution M(μ, κ) on the half circle.

[2] Using the transformation to other kind of polar co-ordinates; for fixed constant c > 0,

$$\{ x_1 \to c\,\mathrm{Tan}[\alpha]\,\mathrm{Cos}[\beta],\ x_2 \to c\,\mathrm{Tan}[\alpha]\,\mathrm{Sin}[\beta] \},$$

we have a joint density density function

$$\mathrm{g22}(\alpha, \beta; \lambda, \mu, \sigma, \rho, \eta, c)$$

$$= \frac{c^2\ \mathrm{Sec}^2\alpha\ \mathrm{Tan}\,\alpha}{2\pi\sqrt{1-\rho^2}\ \sigma^2} *$$

$$\exp\left[-\frac{c^2}{2(1-\rho^2)\sigma^2} * \{(1-\rho\,\mathrm{Sin}\,2\beta)\,\mathrm{Tan}^2\alpha - 2(\mathrm{Cos}(\beta-\mu) - \rho\,\mathrm{Sin}(\beta+\mu))\,\mathrm{Tan}\,\alpha\,\mathrm{Tan}\,\eta + (1-\rho\,\mathrm{Sin}\,2\mu)\,\mathrm{Tan}^2\eta\}\right].$$

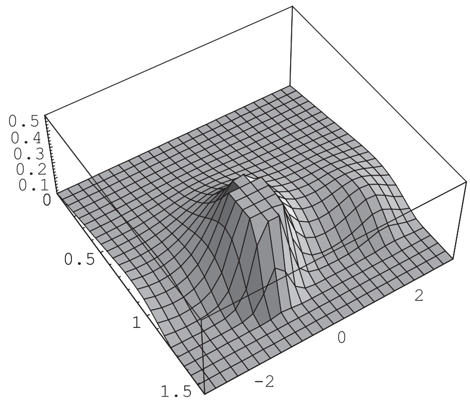

Figure 3.4: The joint density function of g22(α, β ; $\lambda, \mu, \sigma, \rho, \eta$, c) with the parameters {$\sigma\to 1$, c$\to 1$, $\rho\to$ -0.4, $\mu\to$ -π/6, $\eta\to\pi$/6}.

The fieture of this distribution is that it has zero probability at $\alpha = 0$, this is because the density has the factor, $\mathrm{Tan}\,\alpha$.

If we consider the zero mean special case $\eta\to 0$, we can derive the following conditional distributions.

The conditional distribution of the variable α from g22(α, β; $\lambda, \mu, \sigma, \rho$, 0, c) is

$$\text{cond01} = \frac{c^2\, e^{-\frac{c^2 (-1+\rho\,\mathrm{Sin}[2\beta])\,\mathrm{Tan}[\alpha]^2}{2(-1+\rho^2)\sigma^2}}\ \mathrm{Sec}[\alpha]^2 (-1+\rho\,\mathrm{Sin}[2\beta])\,\mathrm{Tan}[\alpha]}{(-1+\rho^2)\,\sigma^2}.$$

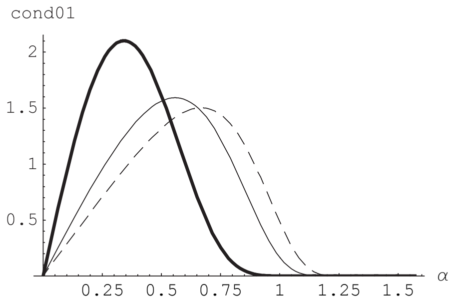

Figure 3.5 : The density of the conditional distribution of α for $\beta\to\pi/4$, c $\to$ 2, $\sigma\to 1$ and $\rho\to\{-0.6, 0, 0.6\}$ (bold, plain, dashed).

The conditional density of the variable β for the joint density g22(α, β ; $\lambda, \mu, \sigma, \rho$, 0, c) is

$$\text{condg02} = \frac{e^{-\frac{c^2\rho\,\mathrm{Cos}[\beta]\,\mathrm{Sin}[\beta]\,\mathrm{Tan}[\alpha]^2}{(-1+\rho^2)\sigma^2}}}{2\pi\,\mathrm{BesselI}\left[0,\ \frac{c^2\rho\,\mathrm{Tan}[\alpha]^2}{2(-1+\rho^2)\sigma^2}\right]},$$

which is the axial version of the von Mises distribution M(0, κ) on the half circle ($-\pi/2 \le \theta \le \pi/2$) when we set $\kappa =$

$\frac{c^2 \rho \operatorname{Tan}[\alpha]^2}{2(-1+\rho^2)\sigma^2}$, and $\rho < 0$.

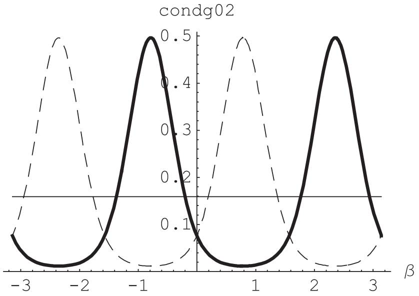

Figure 3.6: The density function of conditional distribution of β
for $\alpha \to \pi/4$, $c \to 1$, $\sigma \to 1$, $\rho \to \{-0.6, 0, 0.6\}$(bold,plain,dashed).

[3] Using the transformation for axial- angular distribution; for fixed constant c > 0,

$\{x_1 \to c \operatorname{Tan}[\alpha],\ x_2 \to c \operatorname{Tan}[\beta]\}$,

we have a joint distribution with a density function

$$g3(\alpha, \beta; c, \sigma, \rho, \mu, \eta) =$$

$$(c^2 \operatorname{Sec}[\alpha]^2 \operatorname{Sec}[\beta]^2)/(2\pi\sigma^2 \sqrt{(1-\rho^2)})*$$

$$\exp(-c^2/(2(1-\rho^2)\sigma^2)\{\operatorname{Tan}^2\alpha + \operatorname{Tan}^2\beta + 2(\rho\operatorname{Cos}\mu - \operatorname{Sin}\mu)\operatorname{Tan}\beta\operatorname{Tan}\eta +$$

$$(1-\rho\operatorname{Sin}2\mu)\operatorname{Tan}^2\eta - 2\operatorname{Tan}\alpha(\rho\operatorname{Tan}\beta + (\operatorname{Cos}\mu - \rho\operatorname{Sin}\mu)\operatorname{Tan}\eta)\}),$$

for $-\pi/2 < \alpha < \pi/2$ and $-\pi/2 < \beta < \pi/2$,
where $\sigma > 0$, $c > 0$, $-1 < \rho < 1$, $-\pi/2 \le \eta < \pi/2$ and $-\pi/2 \le \mu < \pi/2$.

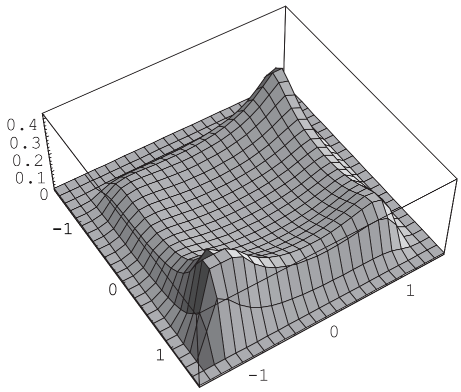

Figure 3.7: The joint density function of $g3(\alpha, \beta ; c, \sigma, \rho, \mu, \eta)$ for $\{\sigma \to 1, c \to \sqrt{1/2}, \rho \to -0.3, \mu \to 0, \eta \to \pi/8\}$.

The conditional distribution of the variable α in $g3(\alpha, \beta; \lambda, \mu, \sigma, \rho, 0, c)$ is

cnd31 =

$$\frac{c\,\mathrm{Sec}[\alpha]^2}{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma} * \exp\Big[-\frac{c^2\{\mathrm{Tan}\,\alpha - \rho\,\mathrm{Tan}\,\beta + (-\mathrm{Cos}\,\mu + \rho\,\mathrm{Sin}\,\mu)\,\mathrm{Tan}\,\eta\}^2}{2(1-\rho^2)\sigma^2}\Big],$$

for $-\pi/2 \le \alpha < \pi/2$, where $\sigma > 0$, $c > 0$, $-1 < \rho < 1$, $-\pi/2 \le \eta < \pi/2$, $-\pi/2 \le \mu < \pi/2$ and $-\pi/2 \le \beta < \pi/2$.

Also the conditional distribution of the variable β in $g3(\alpha, \beta; \lambda, \mu, \sigma, \rho, 0, c)$ is

cnd32 =

$$\frac{c\,\mathrm{Sec}[\beta]^2}{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma} * \exp\Big[-\frac{c^2\{\rho\,\mathrm{Tan}\,\alpha - \mathrm{Tan}\,\beta + (-\rho\,\mathrm{Cos}\,\mu + \mathrm{Sin}\,\mu)\,\mathrm{Tan}\,\eta\}^2}{2(1-\rho^2)\sigma^2}\Big],$$

for $-\pi/2 \le \beta < \pi/2$, where $\sigma > 0$, $c > 0$, $-1 < \rho < 1$, $-\pi/2 \le \eta < \pi/2$, $-\pi/2 \le \mu < \pi/2$ and $-\pi/2 \le \alpha < \pi/2$.

Since the two conditional distributions are identical when $\mathrm{Sin}\,\mu = \mathrm{Cos}\,\mu$, we show the graphs of the conditional distribution of α ;

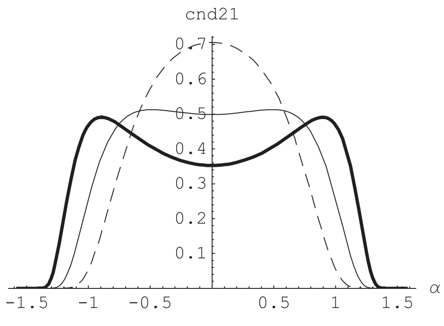

Figure 3.8: The density function of conditional distribution of α for $\sigma \to 1, \rho \to 0.6, \mu \to 0, \eta \to 0, \beta \to 0,\ c \to \{\sqrt{1/2}, 1, \sqrt{2}\}$ (bold, plain, dashed).

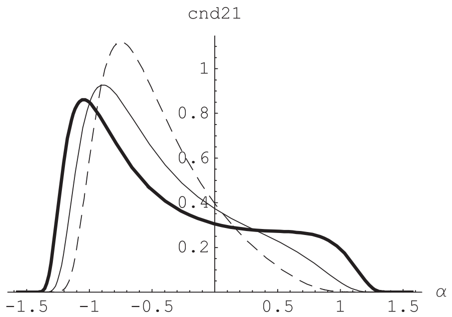

Figure 3.9: The density function of conditional distribution of α for
$\sigma \to 1$, $\rho \to 0.6$, , $\mu \to \pi/3$, $\eta \to \pi/8$, $\beta \to -\pi/4$, $c \to \{\sqrt{1/2}, 1, \sqrt{2}\}$ (bold, plain, dashed).

From these graphs it is seen that the parameter c > 0 of the density function is important since the graph is concave if 0 < c < 1 and also it is convex if c > 1.

4. Some scale mixtures of the distributions on the circle

In this section we consider the extensions of the given distributions by using the scale mixture with several known mixing functions of the scale σ^2 of the multi angular distributions obtained in the section 3. Here we employ the inverse Gamma distribution for the mixing distribution function and apply it to the conditional density functions. Since the mixture distribution seldom has an explicit form, we study the graphical features of the density function of the mixture of distribution.

The inverse Gamma distribution has a probability density function, see Gradshteyn and Ryzhik [2],

$$\mathrm{mf}(x) = \frac{x^{-(a+1)}\, e^{-\frac{1}{b x}}}{\Gamma(a)\, b^a};$$

for $x > 0$, parameters : $a > 0$ (shape), $b > 0$ (scale).

[1] case for the joint distribution g1 of variables β and r ;

In section 3 [1], using the polar transformation, we have obtained the joint distribution g1 of the variables β and r,

$$\mathrm{g1} = \frac{e^{\frac{c^2+r^2-2 c r \mathrm{Cos}[\beta-\mu]-r^2 \rho \mathrm{Sin}[2\beta]+c\rho(-c \mathrm{Sin}[2\mu]+2 r \mathrm{Sin}[\beta+\mu])}{2(-1+\rho^2)x}}\, r}{2\pi\sqrt{1-\rho^2}\, x},$$

where $x = \sigma^2$.

The scale mixture of the distribution with the weight, the inverse Gamma distribution, is given as

$$\text{mxg1}(\text{r}, \beta) = \int_0^{\infty} \text{g1} * \text{mf}(\text{x})\, d\text{x}$$

$$= \frac{1}{\pi\sqrt{1-\rho^2}} 2^{\text{a}} \text{a b r} *$$

$$\{(-1+\rho^2)/(-2 - \text{b}(\text{c}^2+\text{r}^2) + 2\rho^2 + 2\,\text{b c r Cos}[\beta-\mu] +$$

$$\text{b r}^2 \rho\, \text{Sin}[2\beta] + \text{b c}\rho(\text{c Sin}[2\mu] - 2\,\text{r Sin}[\beta+\mu]))\}^{1+\text{a}},$$

for {r, 0, ∞}, {β, $-\pi$, π},
where $\sigma > 0$, $c > 0$, $-1 < \rho < 1$, $-\pi \le \mu < \pi$, $a > 0$ and $b > 0$.

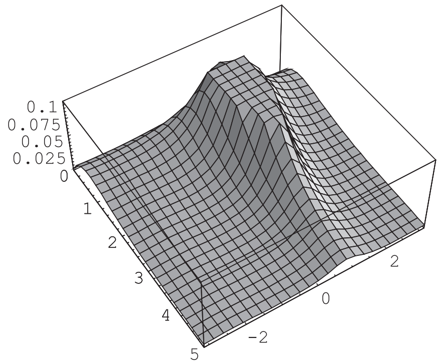

Figure 4.1: The joint density function mxg1(r, β) of the mixture for the parameters {a→1,b→1,c→1, ρ→0.3, μ→π/4}.

This distribution is related to the angular type Student distribution for the circle.
If we set {a→n/2, b→2/n, c→0}, then we can obtain the joint density function of the distribution;

$$\text{mxg10}(\text{r}, \beta) = \frac{\text{r}}{2\pi\sqrt{1-\rho^2}} \left(\frac{\text{n} - \text{n}\rho^2}{\text{n} + \text{r}^2 - \text{n}\rho^2 - \text{r}^2\rho\,\text{Sin}\,2\beta}\right)^{\frac{2+\text{n}}{2}}.$$

The marginal distribution for β of the distribution is given by

$$\int_0^{\infty} \text{mxg10}\, d\text{r} = \frac{\sqrt{1-\rho^2}}{2\pi(1-\rho\,\text{Sin}\,2\beta)}.$$

This distribution may belong to the family of the wrapped Cauchy distributions.

On the other hand the conditional distribution for β has so complicated form that we can not treat it here.

[2] case for the joint distribution g3 of the angular variables α and β;

In section 3 [3], using the different polar transformation, we have obtained the joint distribution g3 of the angular variables α and β,

$$g3 = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma^2} * \exp(D) * c^2\ \mathrm{Sec}[\alpha]^2\ \mathrm{Sec}[\beta]^2\ ,$$

where

$$D = \frac{1}{2(-1+\rho^2)\sigma^2} *$$

$$\{c^2 (\mathrm{Tan}[\alpha]^2 + \mathrm{Tan}[\beta]^2 + 2(\rho\,\mathrm{Cos}[\mu] - \mathrm{Sin}[\mu])\,\mathrm{Tan}[\beta]\,\mathrm{Tan}[\eta] +$$

$$(1 - \rho\,\mathrm{Sin}[2\mu])\,\mathrm{Tan}[\eta]^2 - 2\,\mathrm{Tan}[\alpha]\,(\rho\,\mathrm{Tan}[\beta] + (\mathrm{Cos}[\mu] - \rho\,\mathrm{Sin}[\mu])\,\mathrm{Tan}[\eta]))\}\,.$$

The joint density function of the mixture distribution of g3 is given as

$$\mathrm{mxg3}(\alpha, \beta) =$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} *$$

$$\left(a\,b\,c^2\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sec}[\beta]^2 \left(1 - \frac{1}{2(-1+\rho^2)} (b\,c^2 (\mathrm{Tan}[\alpha]^2 + \mathrm{Tan}[\beta]^2 + 2(\rho\,\mathrm{Cos}[\mu] - \mathrm{Sin}[\mu])\,\mathrm{Tan}[\beta]\,\mathrm{Tan}[\eta] + \right.\right.$$

$$\left.\left.(1 - \rho\,\mathrm{Sin}[2\mu])\,\mathrm{Tan}[\eta]^2 - 2\,\mathrm{Tan}[\alpha]\,(\rho\,\mathrm{Tan}[\beta] + (\mathrm{Cos}[\mu] - \rho\,\mathrm{Sin}[\mu])\,\mathrm{Tan}[\eta])))\right)^{-1-a}\right),$$

for $-\pi/2 < \alpha, \beta < \pi/2$, where $a > 0$, $b > 0$, $c > 0$, $-\pi/2 < \mu, \eta < \pi/2$ and $-1 < \rho < 1$.

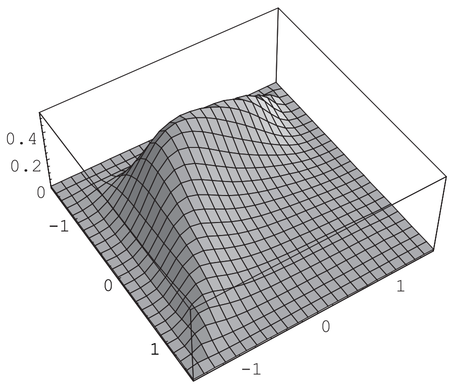

Figure 4.2: The joint density function mxg3 of the mixture with the parameters $\{a\to 2, b\to 1, c\to 1, \rho\to -0.4, \mu\to \pi/6, \eta\to -\pi/6\}$.

Since it is generally difficult to derive the marginal density function and also conditional density function from the mixture distribution mxg3, we here consider a special case $\{\mu\to 0,\ \eta\to 0\}$ and we define the joint density function as mxg30;.

$$\mathrm{mxg30}(\alpha,\beta)=\frac{1}{\pi\sqrt{1-\rho^2}}*2^{\mathrm{a}}\,\mathrm{a}\,\mathrm{b}\,\mathrm{c}^2\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sec}[\beta]^2 * \left(\frac{-1+\rho^2}{2(-1+\rho^2)-\mathrm{b}\,\mathrm{c}^2(\mathrm{Tan}[\alpha]^2-2\rho\,\mathrm{Tan}[\alpha]\,\mathrm{Tan}[\beta]+\mathrm{Tan}[\beta]^2)}\right)^{1+\mathrm{a}},$$

for $-\pi/2<\alpha<\pi/2$, $-\pi/2<\beta<\pi/2$, where a > 0, b > 0, c > 0 and $-1<\rho<1$.

It is known that if we set { a → n/2, b → 2/n } in the normal scale mixture distribution on the line with the mixing density function mf (the density of the inverse Gamma distribution), then the mixture is Student's t-distribution with degree of freedom n. Thus we consider the case when { a → n/2, b → 2/n } in the joint density function mxg30.

Setting {a → n/2, b → 2/n } in mxg30, we have the joint density function of α and β;

$$\mathrm{mxg30n}(\alpha,\beta)=\frac{1}{\pi\sqrt{1-\rho^2}}*2^{\mathrm{n}/2}\,\mathrm{c}^2\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sec}[\beta]^2 * \left(\frac{-1+\rho^2}{2(-1+\rho^2)-\frac{2\mathrm{c}^2(\mathrm{Tan}[\alpha]^2-2\rho\,\mathrm{Tan}[\alpha]\,\mathrm{Tan}[\beta]+\mathrm{Tan}[\beta]^2)}{\mathrm{n}}}\right)^{1+\frac{\mathrm{n}}{2}},$$

for $-\pi/2<\alpha<\pi/2$, $-\pi/2<\beta<\pi/2$, where n > 0, c > 0 and $-1<\rho<1$.

This angular distribution mxg30n is very interesting because it has many different shapes with the parameters n, ρ and c. The Graphs of this joint density are shown in Figures 4.3 - 4.8 below. In particular, the parameter c > 0 seems to play a key point.

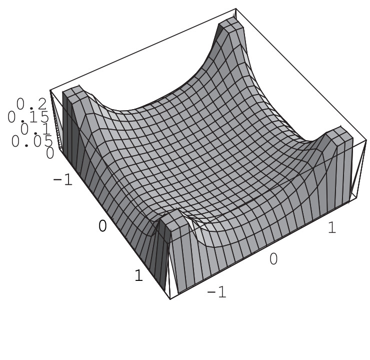

Figure 4.3: The joint density function mxg30n with the parameters {n→1,c→1/2,ρ→-0.0}.

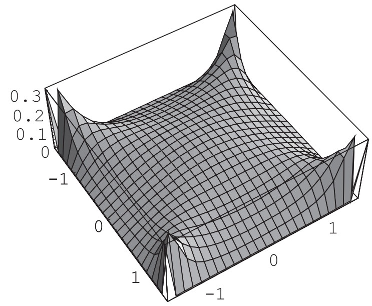

Figure 4.4: The joint density function mxg30n with the parameters {n→1,c→1,ρ→-0.0}.

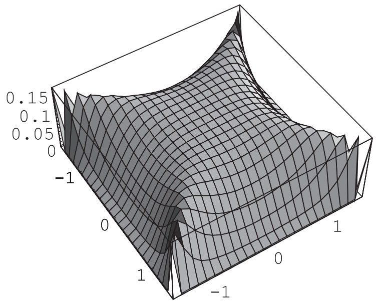

Figure 4.5: The joint density function mxg30n with the parameters {n→2, c→1, ρ→ 0.0}.

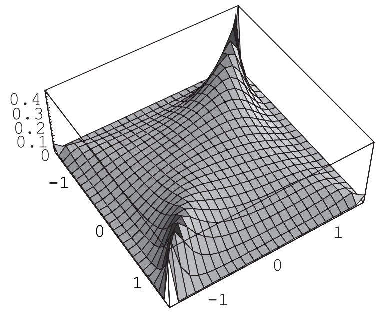

Figure 4.6: The joint density function mxg30n with the parameters {n→2,c→1,ρ→-0.5}.

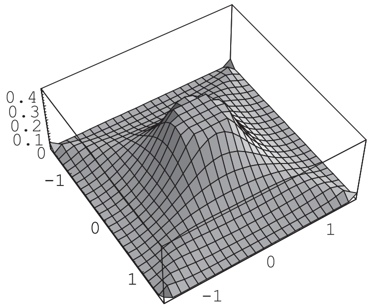

Figure 4.7: The joint density function mxg30n with the parameters
{n→2,c→2,ρ→0.0}.

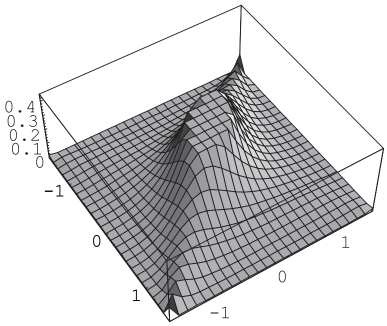

Figure 4.8: The joint density function mxg30n with the parameters
{n→2, c→2, ρ→-0.5}.

When n = 1 and c = 1 in mxg30n, we have the joint density of α and β,

$$\frac{\sqrt{2}\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sec}[\beta]^2\left(\frac{-1+\rho^2}{2(-1+\rho^2)-2(\mathrm{Tan}[\alpha]^2-2\rho\,\mathrm{Tan}[\alpha]\,\mathrm{Tan}[\beta]+\mathrm{Tan}[\beta]^2)}\right)^{3/2}}{\pi\sqrt{1-\rho^2}}.$$

It is seen that its marginal distribution of α is the uniform distribution on the half circle.

Also when n = 1 in mxg30n, its marginal distribution of α is

$$\mathrm{mt1}(\alpha) = \frac{1}{2\pi}\,\frac{4\,\mathrm{c}}{(1+\mathrm{c}^2)+(1-\mathrm{c}^2)\,\mathrm{Cos}[2\,\alpha]},$$

for $-\pi/2 < \alpha < \pi/2$, $c > 0$.

It should be notable that this marginal distribution is equivalent to the wrapped Cauchy distribution WC(μ, ρ) for $\theta = 2\alpha$ given in section 2 if we set $\mu = 0$, $\rho = -(1 - c^2)/2$ for $c^2 < 3$. Therefore the distribution with the density mt1 is a generalization of the wrapped Cauchy distribution WC(μ, ρ) for $\theta = 2\alpha$ with parameter c > 0.

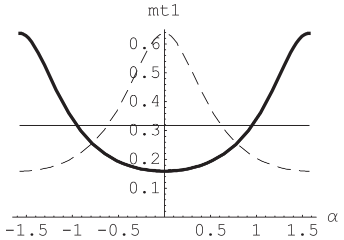

Figure 4.9 : The density of the marginal distribution of α with n = 1 for c → {1/2, 1, 2} (bold, plain, dashed).

If we set n = 2 in mxg30n(α, β), its marginal density of α is given by

$$\frac{2\sqrt{2}\ c\,\mathrm{Cos}[\alpha]}{(2 + c^2 - (-2 + c^2)\,\mathrm{Cos}[2\,\alpha])^{3/2}},\quad \text{for} -\pi/2 < \alpha < \pi/2,\ c > 0.$$

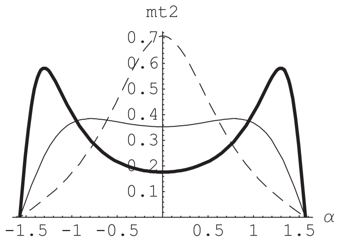

Figure 4.10 : The density of the marginal distribution of α with n = 2 for c → {1/2, 1, 2} (bold, plain, dashed).

Finally, we consider the limiting behavior of the density mxg30n(α,β) in the parameter n. In the limit of large n the joint density of the distribution becomes

$$\frac{c^2\, e^{-\frac{c^2 (\mathrm{Tan}[\alpha]^2 - 2\rho\,\mathrm{Tan}[\alpha]\,\mathrm{Tan}[\beta] + \mathrm{Tan}[\beta]^2)}{2(1-\rho^2)}}\ \mathrm{Sec}[\alpha]^2\ \mathrm{Sec}[\beta]^2}{2\pi\sqrt{1-\rho^2}}.$$

This joint density will be related to that of the bivariate normal distribution on the line.

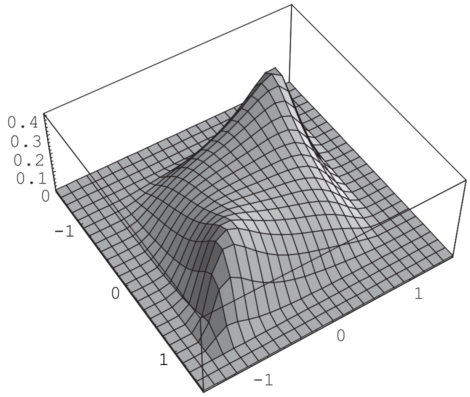

Figure 4.11: The joint density function mxg30n with the parameters $\{n \to \infty, c \to 1.2, \rho \to -0.5\}$.

5. Derivation of a distribution from the angular distribution

We consider a derivation of a distribution on the plane by using the inverse transformation to the angular distributions obtained in section 3. Here we consider the conditional distribution of α in the joint distribution with the density g3;

$$\text{cond3}\,(\alpha;\, c^2) = \frac{c\, e^{\frac{c^2 (\text{Tan}[\alpha]-\rho\,\text{Tan}[\beta]+(-\text{Cos}[\mu]+\rho\,\text{Sin}[\mu])\,\text{Tan}[\eta])^2}{2\,x\,(-1+\rho^2)}}\ \text{Sec}[\alpha]^2}{\sqrt{2\pi}\,\sqrt{x}\,\sqrt{1-\rho^2}}.$$

Applying the same method for the scale mixture to the conditional distribution (cond3) concerning the parameter c^2 in stead of σ^2 discussed in section 4 [3], we can obtain the scale mixture distribution of the conditional distribution (cond3) with the density function;

$$\text{mxc3}\,(\alpha;\ \beta) = \int_0^\infty \text{cond3}\,(\alpha;\, \omega)\,\text{mf}\,(\omega)\, d\omega$$

$$= \frac{2^{\frac{3}{4}-\frac{a}{2}}\, b^{-a}}{\sqrt{\pi}\,\sigma\,\sqrt{1-\rho^2}\,\Gamma[a]} * \text{BesselK}\Big[-\frac{1}{2}+a,\ \sqrt{2}\,\sqrt{-\frac{(\text{Tan}[\alpha]-\rho\,\text{Tan}[\beta]+(-\text{Cos}[\mu]+\rho\,\text{Sin}[\mu])\,\text{Tan}[\eta])^2}{b\,\sigma^2\,(-1+\rho^2)}}\ \Big] *$$

$$\text{Sec}[\alpha]^2 \left(-\frac{b\,(\text{Tan}[\alpha]-\rho\,\text{Tan}[\beta]+(-\text{Cos}[\mu]+\rho\,\text{Sin}[\mu])\,\text{Tan}[\eta])^2}{\sigma^2\,(-1+\rho^2)}\right)^{\frac{1}{4}(-1+2\,a)},$$

for $-\pi/2 < \alpha < \pi/2$, where $-\pi/2 \le \eta < \pi/2$, $-\pi/2 \le \mu < \pi/2$, $-\pi/2 \le \beta < \pi/2$, $a > 0$, $b > 0$, $c > 0$ and $-1 < \rho < 1$.

Using a inverse transformation;

$$\{\alpha \to \text{ArcTan}[x_1],\ \beta \to \text{ArcTan}[x_2]\},$$

to the angular conditional distribution (cond3), we can obtain a conditional distribution of x_1 given x_2 with the density function;

$$\varphi(x_1 ; \mu_1, \mu_2, \sigma, \rho, a, b, x_2) = \left(2^{\frac{3}{4}-\frac{a}{2}} b^{-a} \mathrm{BesselK}\left[-\frac{1}{2}+a, \sqrt{2}\, \frac{\mathrm{Abs}\left[x_1 - \rho x_2 + \frac{(-1+\rho\mu_1)\mu_2}{\sqrt{1+\mu_1^2}}\right]}{\sqrt{b(1-\rho^2)\sigma^2}}\right] \left(\frac{b\left(x_1 - \rho x_2 + \frac{(-1+\rho\mu_1)\mu_2}{\sqrt{1+\mu_1^2}}\right)^2}{(1-\rho^2)\sigma^2} \right)^{\frac{1}{4}(-1+2a)} \right) \Big/ \left(\sqrt{\pi - \pi\rho^2}\, \sigma\, \Gamma[a] (1 + x_2^2) \right),$$

for $-\infty < x_1 < \infty$, where $a > 1$, $b > 0$, $\sigma > 0$, $-1 < \rho < 1$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$ and $-\infty < x_2 < \infty$.

Unfortunately, this distribution will belong to the generalized asymmetric Laplace distribution (or the Bessel K-function distribution) on the line, see Kotz, Kozubowski and Podgorski [5].
Figure 5.1 and Figure 5.2 show graphs of its distribution.

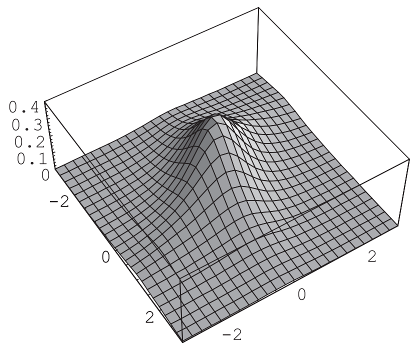

Figure 5.1: The density function of $\varphi(x_1 ; \mu_1, \mu_2, \sigma, \rho, a, b, x_2)$ with the parameters {$a\to 2$, $b\to 1$, $\sigma\to 1$, $\rho\to -0.5$, $\mu_1\to -0.5$, $\mu_2\to -0.5$}.

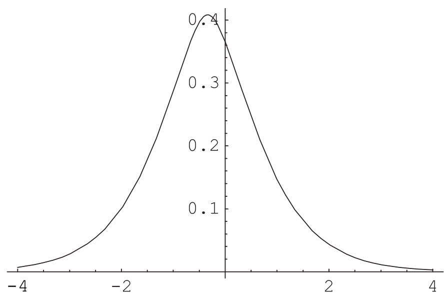

Figure 5.2: The density function of conditional distribution φ of X_1 when {$X_2\to 0$, $a\to 2$, $b\to 1$, $\sigma\to 1$, $\rho\to -0.5$, $\mu_1\to -0.5$, $\mu_2\to -0.5$}.

Conclusion

In this paper, we have discussed a problem of how to construct angular distributions considering the generalization of the well-known von Mises distributions on the circle. In order to construct the distributions we have introduced three methods of the transformations to polar co-ordinates from a bivariate normal distribution instead of the radial projection of the distribution (which is the well known case and leads to the projected normal distribution). Furthermore, we considered the scale mixture of the obtained distributions with the inverse Gamma distribution for the mixing distribution. The obtained density functions have very complex mathematical forms, and it makes the study quite difficult. Nevertheless, certain basic distributions on the circle have been derived. Since a distribution on the line has a corresponded circular distribution, it seems to be possible to derive a new distribution on the line from a new angular distribution by using the inverse transformation. In section 5, we have introduced an example and obtained a distribution on the line from a new angular distribution. It is unfortunate that this distribution belongs to a known distribution family. The further study of this problem must involve the consideration of the theoretical property of the obtained distributions. It must be the future work to discuss the point estimation of the parameters of distributions and their application to axial data.

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