Asymptotic order of the expected length of excursions for the processes with a Scale mixture of normal distribution, II

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Abstract. In this paper we provide a proof of the conjecture for the asymptotic order of the expected mean length of excursions for some ellipsoidal processes with a scale mixture of normal (SMN) distribution whose mixing distribution is the generalized Gamma distribution, which was presented in the previous paper of Tanaka [15]. It is seen that the L'Hopital's rule and the Abelian theorem for the two-sided Laplace transforms are useful to evaluate the asymptotic order in the proof.

Keywords: *Abelian theorem, generalized Gamma distribution*, *Laplace transform, length of excursions, Scale mixture of normal distribution*

1. Introduction

For the zero-mean stationary Gaussian process X_t with autocovariance γ(h) in continuous time the order of the expected mean length of excursions above level u is given by $O(u^{-1})$ when u is sufficiently large (see, for example, Kedem [8], p.138). It is also seen that in the process having the Pearson Type VII distribution its order of the length of excursions in terms of u is $O(u^0)$, that is, constant in u (Tanaka and Shimizu [14]). There are papers extended the results to other stationary ellipsoidal processes which have the

Laplace distribution, the generalized Laplace distribution and the Logistic distributions. In the previous paper of Tanaka [15], we have given a conjecture for the case of the stationary ellipsoidal process with a generalized Gamma distribution as the mixing density function, which is an extension of the case for the generalized Laplace distribution; the order of the expected mean length of excursions above level u will be given *by* $O\left(u^{-\frac{\gamma}{1+\gamma}}\right)$, where γ is a scale parameter of the generalized Gamma distribution. As the special cases we can derive that if we set $\gamma \to 0$, then we have the case of the Pearson Type VII distribution, $O(u^{-0}) = O(1)$; if $\gamma = 1$, the cases of the generalized Laplace distribution and the logistic distribution, $O(u^{-1/2})$; and if $\gamma \to \infty$, then the case of the Normal distribution, $O(u^{-1})$.

The objective of this paper is to provide a proof of the conjecture. The L'Hopital's rule and the Abelian theorem for the two-sided Laplace transforms are useful to evaluate the asymptotic order in the proof.

2. Definition and Notations

Following the previous paper (Tanaka [15]), we shall suppose throughout that $\{X(t)\}$ is a stationary zero-mean and unit-variance ellipsoidal process with the probability density function $f(x)$ and the autocorrelation function ρ(h) which is twice differentiable at $h = 0$. An expected mean length of excursions above level u discussed in Tanaka and Shimizu ([12], [14]) and Tanaka ([13], [15]) for the discrete time ellipsoidal process is the following ratio of the two integrals:

$$I_u(\mathrm{N}) = \frac{\int_u^\infty f(x)\,dx}{\frac{1}{\pi\Delta}\int_\rho^1 \frac{1}{\sqrt{1-t^2}}\int_0^\infty f\left(\sqrt{x^2+\frac{2u^2}{1+t}}\right)dx\,dt}. \quad (2.1)$$

The continuous time formula of (2.1) is also given as

$$\lim_{N\to\infty} I_u(N) = \frac{\pi\int_u^\infty f(x)\,dx}{\sqrt{-\rho^{(2)}(0)}\int_0^\infty f\left(\sqrt{x^2+u^2}\right)dx} \quad (2.2)$$

$$= A(u), \quad say.$$

Particularly when *f(x)* is a standard normal *N(0,1)* density and Φ is the distribution function, we have the well-known result such that

$$A(u) = \frac{2\pi\,[1-\Phi(u)]}{\sqrt{-\rho^{(2)}(0)}\,exp\{-\frac{1}{2}u^2\}} \tag{2.3}$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\,\frac{1}{u} \qquad as\ u \to \infty.$$

(see (a) in Problem 15 of Kedem [10]). Then it is interesting to estimate the order of smallness of *A(u)* when $u\to\infty$ for ellipsoidal processes with a non-Gaussian distribution function *f(x)*.

Let $H(u)$ be a power function of u with a negative order, i.e. $H(u) = C_0\,u^{-\alpha}$ for $\alpha > 0$, $C_0 > 0$. If a function $G(u) \sim H(u)$ as $u \to \infty$, then $H(u)$ is called the asymptotic order function of *G(u)* with the order $(-\alpha)$ when $u \to\infty$. For example, from (2.3) the asymptotic order function of *A(u)* for the Gaussian process is

$$H(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\,\frac{1}{u} \tag{2.4}$$

and its order is *(-1)*.

To consider the limit of the ratio in (2.2) when $u\to\infty$, we introduce the completely monotonic functions, because almost ellipsoidal density functions are completely monotonic (see Andrews and Mallows [1]). We say that the function *f(x)* is completely monotonic in $[0, \infty)$ if it satisfies $(-1)^k f^{(k)}(x) \geq 0$ for $0 \leq x < \infty$. Bernstein's theorem (see Widder [17], Theorem 19-b) shows that if *f(x)* is completely monotonic, *f(x)* is expressed as the Laplace transform of some function $\alpha(t)$ such that

$$f(x) = \int_0^\infty e^{-xt}\,d\alpha(t)$$

$$= \int_0^\infty e^{-xt}\,\theta(t)\,dt, \tag{2.5}$$

where α(t) is bounded and non-decreasing in $[0, \infty)$ and absolutely continuous, i.e. $d\,\alpha(t) = \theta(t)\,dt$. Furthermore the density function *f(x)* in (1.5) can be also expressed by *a scale mixture of normal distribution (SMN)* with a mixing function *G(s)*, or Mellin-Stieltjes transform of *G(s)* with the kernel *N(0,1)*, i.e.

$$f(\tau) = \int_0^\infty \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x^2}{2s}}\, dG(s)$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x^2}{2s}}\, g(s)\, ds, \qquad (2.6)$$

where $\tau = \frac{x^2}{2}$ and *G(s)* is an absolutely continuous and $d\,G(s) = g(s)\,ds$. From (2.5) and (2.6) we have

$$\theta(t) = \frac{t^{-3/2}}{\sqrt{2\pi}} g\left(\frac{1}{t}\right) . \qquad (2.7)$$

For example, in the case of the Gaussian process (when $f(\tau)$ is the density of standard normal distribution), we have $G(s) = U(s-1) = 0$ ($s \le 1$), $= 1$ ($s > 1$), and then $g(s) = \delta(s-1)$ (*Dirac's delta* function).

Using the well-known L'Hospital's rule (see Hardy [8] and Tichmarsh [16]), we can derive the following lemma (see Tanaka [15]).

Lemma A. *Let p(x), q(x) and r(x) be the real-valued continuous and differentiable functions on some neighborhood $[a, \infty)$ of infinity. Suppose that $r(x)\, q(x) \neq 0$, $q'(x) \neq 0$ for $x \in [a, \infty)$, and $\lim_{x\to\infty} p(x) = 0$, $\lim_{x\to\infty} r(x)\, q(x) = 0$.*

If (i) there exists constant $C_1 > 0$ such that $\lim_{x\to\infty} \dfrac{p'(x)}{r(x)\, q'(x)} = C_1$,

i.e. $\dfrac{p'(x)}{q'(x)} \sim C_1\, r(x)$ *as* $x \to \infty$,

(ii) $\lim_{x\to\infty} \dfrac{r'(x)\, q(x)}{r(x)\, q'(x)} = C_3$, *where C_3 is a constant,*

then there exists a constant $C_2 > 0$ such that $\lim_{x\to\infty} \dfrac{p(x)}{r(x)\, q(x)} = C_2$,

i.e. $\dfrac{p(x)}{q(x)} \sim C_2\, r(x)$ *as* $x \to \infty$, *where* $C_2 = \dfrac{C_1}{1+C_3}$. *In addition, if* $C_3 = 0$ *in the condition (ii), then*

$$\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)} \quad as\ x \to \infty.$$

Corollary A. *Suppose that* $r(x) = x^{\alpha}$ $(\alpha \neq 0)$ *in Lemma A. If q(x) satisfies the condition (i) and*

$$(ii)' \qquad lim_{x\to\infty} \frac{q(x)}{x\,q'(x)} = 0,$$

then

$$\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)} \quad as\ x \to \infty.$$

Note that if $r(x) = x^{\alpha}$ $(\alpha = 0)$, the condition (ii) of Lemma A is always holds.

Applying Lemma A (or Corollary A) to the ratio *A(u)* of (2.2), the partial differentiation with respect to *u* and taking the limit as $u \to \infty$ will lead the following result (see Tanaka [15]).

Theorem A. *Suppose that f(x) is a scale mixture of normal distribution (SMN) with a mixing density function g(a) and it is expressed by (2.6). Let A(u) be the expected mean length of excursions above a level u in (2.2) such that, for some d≥0,*

$$A(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \left(\frac{\int_u^{\infty} f(x)\,dx}{\int_0^{\infty} e^{-\frac{u^2}{2s}}\, g(s)\,ds} \right) \qquad (2.8)$$

$$= O(u^{-d}),$$

as $u\to\infty$. *If* $d > 0$ *and g(s) satisfies either*

$$lim_{u\to\infty} \left(\frac{\int_0^{\infty} e^{-\frac{u^2}{2s}}\, g(s)\,ds}{u^2 \int_0^{\infty} \frac{1}{s}\, e^{-\frac{u^2}{2s}}\, g(s)\,ds} \right) = 0, \qquad (2.9)$$

or

$$lim_{u\to\infty}\left(\frac{\int_u^{\infty} f(x)\,dx}{u\ f(u)}\right) = 0, \qquad (2.9)'$$

then, as u →∞,

$$A(u) \sim \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{f(x)}{u\int_0^{\infty}\frac{1}{s}\,e^{-\frac{u^2}{2s}}\,g(s)\,ds}\right) \qquad (2.10)$$

$$= B(u)\ ,\ say.$$

Furthermore if A(u) = O(1), then (2.3) always holds.

We should note that these results are modified versions of the previous results given by the author (Lemma and Theorem in Tanaka [13]). The ratio *B(u)* in (2.10) can be expressed in terms of the Laplace transforms such that

$$B(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{\int_0^{\infty} e^{-\left(\frac{u^2}{2}\right)t}\,\theta_1(t)\,dt}{u\int_0^{\infty} e^{-\left(\frac{u^2}{2}\right)t}\,\theta_2(t)\,dt}\right), \qquad (2.11)$$

where $\theta_1(t) = \frac{t^{-3/2}}{\sqrt{2\pi}}\,g(t^{-1})$ and $\theta_2(t) = t^{-1}\,g(t^{-1})$.

From the Laplace transform formula (2.11) we may evaluate the asymptotic order of *A(u)* of the scale mixture of normal distribution *f(x)* by using the well-known Abelian theorem for the Laplace transform (see, for example, Bingham, Goldie and Teugels [2], Widder [17]). When *f(x)* is a normal density function, *B(u)* is simpler than *A(u)* and it is easily seen that *B(u)* is of order 1/u. We can also use Theorem 1 for the evaluations of the asymptotic order of *A(u)* for some non Gaussian distribution, such as the generalized Laplace distribution, the Pearson Type VII distribution, the Logistic distribution and the inverse Gauss distribution as a mixing function of the *SMN* (see Examples 1 to 5 in Tanaka [15]).

3. Main Results

We now consider the mixing density function such that the density of the *generalized Gamma* distribution, for $\alpha > 0, \beta > 0, \gamma > 0$,

$$g_0(x) = \frac{\gamma x^{\alpha-1}}{\text{Gamma}[\alpha/\gamma]\, \beta^{\alpha}}\, e^{-(x/\beta)^{\gamma}} \qquad (x > 0) \qquad (3.1)$$

(see Johnson, Kotz and Balakrishnan [9], a Gamma distribution is the special case when $\gamma = 1$). It is seen that the mixing density of the Pearson Type VII distribution may be asymptotically equivalent to that of (3.1) when $\gamma \to 0$. The generalized Laplace, the logistic and the density function of the inverse Gauss distribution have the same kind mixing function of (3.1) with $\gamma = 1$. Also when $\gamma \to \infty$, the mixing function of (3.1) will be $\delta(a-1)$ (Dirac's delta function) and then this asymptotically corresponds to the case for the standard normal distribution. However it is notable that the general formulas of $A(u)$ in (2.3) or $B(u)$ in (2.4) for this mixing density are not expressed by the simple functions and are very difficult to estimate their asymptotic orders directly. For example, if we set $\alpha = 2$, $\beta = 1$ and $\gamma = 2$, then the *SMN* density function is given by

$$\frac{1}{3\sqrt{2\pi}} \Big(3\,\Gamma[\tfrac{3}{4}]\, HypergeometricPFQ\big[\{\}, \{\tfrac{1}{4}, \tfrac{1}{2}\}, -\tfrac{x^4}{16}\big] +$$
$$2x^2 \Big(-3\,\Gamma[\tfrac{5}{4}]\, HypergeometricPFQ\big[\{\}, \{\tfrac{3}{4}, \tfrac{3}{2}\}, -\tfrac{x^4}{16}\big] +$$
$$\sqrt{2\pi}\; x\, HypergeometricPFQ\big[\{\}, \{\tfrac{5}{4}, \tfrac{7}{4}\}, -\tfrac{x^4}{16}\big]\, Sign[x]\Big)\Big),$$

where *HypergeometricPFQ[.]* is the generalized hypergeometric function ${}_pF_q(\boldsymbol{a}; \boldsymbol{b}; x)$ (see Gradshteyn and Ryzhik [8]). The graphs of the mixing density function (the *generalized Gamma* distribution) and its *SMN* distribution density function are shown in Figure 1 and Figure 2.

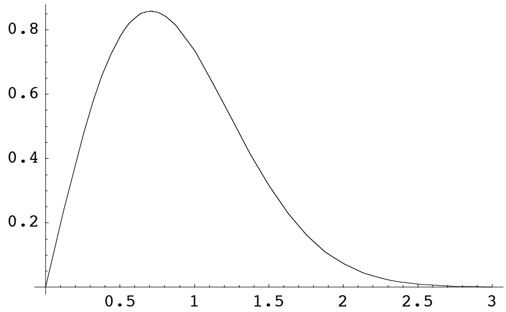

Figure 1. Graph of the mixing density function, the g*eneralized Gamma* distribution with $\alpha = 2, \beta = 1$ and $\gamma = 2$.

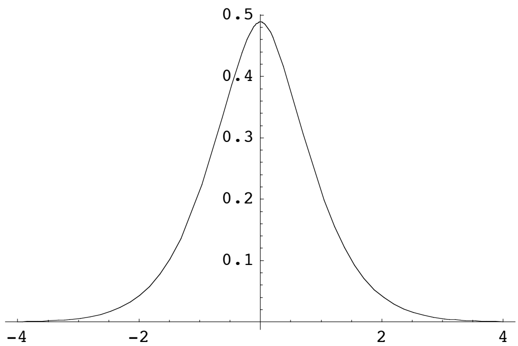

Figure 2. The graph of the density function of the *SMN* distribution with parameters $\alpha = 2, \beta = 1$ and $\gamma = 2$.

By the way, using Theorem A above and an Abelian theorem for two-sided Laplace transforms due to Balkema, Kluppelberg and Resnick [3], we can prove the following main theorem, which was the conjecture given in Tanaka [15] for the asymptotic order function of the process with the generalized Gamma distribution as the mixing function.

Theorem. *Let the process* X_t *have the SMN (scale mixture of normal) distribution f(x) with the mixing density function* $g(s) \sim g_0(s)$, *as* $s \to \infty$, *where* g_0 *is a density function of the generalized Gamma distribution of (3.1). Then the asymptotic order function H(u) of A(u) in (2.2) is given by, for* $\beta > 0$ *and* $\gamma > 0$,

$$H(u) = \frac{\sqrt{2\pi}}{\sqrt{-\rho^{(2)}(0)}} \left(\frac{1}{2\gamma\beta^{\gamma}}\right)^{\frac{1}{2(\gamma+1)}} u^{-\frac{\gamma}{\gamma+1}} . \qquad (3.2)$$

A proof of this theorem will be given in Appendix below. As the special cases of (3.2) we have the following.

Corollary. *In (3.2) of Theorem, asymptotically we have*

(i) when $\gamma = 1$, $H(u) = O(u^{-1/2})$,

(ii) when $\gamma \to \infty$, $H(u) = O(u^{-1})$.

We note that the result (i) of Corollary corresponds to the cases of the generalized Laplace distribution and the logistic distribution, and also (ii) is the case of the Normal distribution. By the way, if we set $\gamma \to 0$ in (3.2), the asymptotic order will tends to zero, but the constant term $\left(\frac{1}{2\gamma\beta^{\gamma}}\right)^{\frac{1}{2(\gamma+1)}}$ will not be bounded. Hence we can not directly obtain the case of the Pearson Type VII distribution from (3.2) in Theorem.

Appendix

Proof of Theorem

From Theorem A, in order to evaluate *A(u)* asymptotically we may consider *B(u)* in (2.11) which has two Laplace transforms, the part of the numerator and that of the denominator. So we may use the Abelian theorem for two-sided Laplace transforms due to Balkema, Kluppelberg and Stadtmulle [4]. We shall give some preliminaries about self-neglecting functions which are needed in their theorem.

A function *s(t)* defined on a left neighborhood of a point t_∞ is self-neglecting (or Beurling slowly varying) if it is strictly positive and satisfies

$$s(t + x\,s(t))/s(t) \to 1 \text{ as } t \to t_\infty, \qquad (\text{A. } 1)$$

uniformly on the bounded *x* intervals.

Note that if $t_\infty < \infty$ and both s and s' vanish at t_∞, then s is self-neglecting. See Balkema, Kluppelberg and Resnick [3] for further information.

The function ψ(t) is an asymptotically parabolic function if $\psi^{(2)}(t)$ exists and is continuous and positive and also satisfies that

$$s(t) = \frac{1}{\sqrt{\psi^{(2)}(t)}}$$

is self-neglecting. In this case *s(t)* is called the scale function of ψ.

Let ψ be asymptotically parabolic with scale function s. Then a positive function *h(t)* is flat for ψ if it satisfies

$$h(t + x\, s(t)) / h(t) \rightarrow 1 \text{ as } t \rightarrow t_\infty, \qquad \text{(A. 2)}$$

uniformly on the bounded x intervals (See [4], p.387).

The conjugate transform of ψ is defined by

$$\psi^*(\xi) = \operatorname*{Sup}_x(\xi \mathrm{x} - \psi(x)). \qquad \text{(A. 3)}$$

Here we consider an integrable nonnegative function g on the real line with a very thin upper tail in the sense that

$$\mathrm{g(t)} > 0 \quad (\mathrm{t} > t_0, \text{ for some } t_0), \qquad \text{(A. 4)}$$

$$\mathrm{g(t)}\, e^{nt} \rightarrow 0 \quad \text{as t} \rightarrow \infty. \qquad \text{(A. 5)}$$

Theorem B*. Let g(t) satisfy (A. 4) and (A. 5). If* $g(t) \sim h(t)\, e^{-\psi(t)}$ *(as t→* t_∞*), where* ψ*(t) is asymptotically parabolic and h(t) is flat for* ψ*, then the two-sided Laplace transform of g(t) satisfies*

$$\int_{-\infty}^{\infty} e^{\tau t}\, g(t)\, dt \sim \beta(\tau)\, e^{\psi^*(\tau)} \quad (\text{as } \tau \rightarrow \infty), \qquad \text{(A. 6)}$$

where ψ^* *is convex conjugate of* ψ*,* β *is flat for* ψ^* *and*

$$\beta(\tau) = \sqrt{2}\, s(t)\, h(t) \qquad \text{(A. 7)}$$

with $s(t)$ is the scale function of ψ, and τ and t are conjugate variables with $\tau = \psi'(t)$ and $t = (\psi^*)'(\tau)$.

(See Balkema, Kluppelberg and Stadtmuller [4], Theorem C).

Now we assume that the process X_t has the *SMN* distribution $f(x)$ with the mixing density function $g_0(s)$ which is the generalized Gamma distribution given in (3.1). Then we have

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} g_0(s)\, ds, \qquad \text{(A. 8)}$$

The two-sided Laplace formula expression for $f(u)$ is given as, putting $\tau = \frac{u^2}{2}$,

$$f(u) = \int_{-\infty}^\infty e^{\tau t} \theta_1(t)\, dt,$$

where $\theta_1(t) = 0$ when $t \geq 0$, and when $t < 0$,

$$\theta_1(t) = \frac{1}{\sqrt{2\pi}} (-t)^{-\frac{3}{2}} g_0(-1/t) \qquad \text{(A. 9)}$$

$$= \frac{1}{\sqrt{2\pi}} (-t)^{-3/2} \frac{\gamma(-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\beta^\gamma} e^{-(-t/\beta)^{-\gamma}}$$

$$(\alpha > 0,\ \beta > 0,\ \gamma > 0).$$

Hence we can set

$$\theta_1(t) = h_1(t) e^{-\psi(t)}, \qquad \text{(A. 10)}$$

where $h_1(t) = \frac{1}{\sqrt{2\pi}} (-t)^{-3/2} \frac{\gamma(-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\beta^\gamma}$,

and $\psi(t) = (-t/\beta)^{-\gamma} \qquad (t < 0).$

Then it is seen that $\theta_1(t)$ is log-concave and its two-sided Laplace transform $L\theta_1(t) = f(u)$ has a nondegenerate interval of existence.

In this case the point t_∞ is zero, and then we can show that ψ(t) is asymptotically parabolic and also h(t) is flat for ψ.

For if we set $s(t) = 1 \big/ \sqrt{\psi''(t)}$, then $s(t)$ must be self-neglecting. This is because that both

$$s(t) = \frac{1}{\sqrt{\beta^\gamma \gamma (1+\gamma)}} (-t)^{1+\gamma/2} \qquad \text{(A. 11)}$$

and the derivative

$$s'(t) = -\frac{(-t)^{\gamma/2} (2+\gamma)}{2\sqrt{\beta^\gamma \gamma (1+\gamma)}} \qquad \text{(A. 12)}$$

vanish at $t_\infty = 0$ for all $\beta > 0$ and $\gamma > 0$. Therefore ψ(t) is asymptotically parabolic. Also we see that $h(t)$ is flat for ψ, since

$$\frac{h(t + x\, s(t))}{h(t)} = \left(1 - \frac{(-t)^{\gamma/2}\, x}{\sqrt{\beta^\gamma \gamma (1+\gamma)}}\right)^{-\frac{1}{2}-\alpha}. \qquad \text{(A. 13)}$$

Therefore the conditions of Theorem B holds, and then we can apply Theorem B to the density function of the scale mixture of normal distribution whose mixing distribution is the generalized Gamma distribution. Thus from (A. 6) in Theorem B, we have

$$f(u) = \int_{-\infty}^{\infty} e^{\tau t}\, \theta_1(t)\, dt \sim \beta_1(\tau)\, e^{\psi^*(\tau)}, \qquad \text{(A. 14)}$$

where

$$\beta_1(\tau) = \sqrt{2}\, s(t)\, h_1(t)$$

$$= \sqrt{2} \left\{ \frac{1}{\sqrt{\beta^\gamma \gamma (1+\gamma)}} (-t)^{1+\gamma/2} \right\} \left\{ \frac{1}{\sqrt{2\pi}} (-t)^{-3/2} \frac{\gamma (-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\, \beta^\gamma} \right\}$$

$$= \frac{\gamma^2 (1+\gamma)}{\sqrt{\pi}\, (\beta^\gamma \gamma (1+\gamma))^{3/2}\, \Gamma[\frac{\alpha}{\gamma}]} (-t)^{\frac{1}{2}(1-2\alpha+\gamma)}. \qquad \text{(A. 15)}$$

and

$$\psi^* (\tau) = (-1)(\gamma + 1)\left(\frac{\beta\tau}{\gamma}\right)^{\frac{\gamma}{\gamma+1}}. \qquad \text{(A. 16)}$$

This is because that from (A. 3),

$$\psi^*(\xi) = \operatorname*{Sup}_x(\xi x - \psi(x)) = \xi x^* - \psi(x^*), \qquad \text{(A. 17)}$$

$$\text{where } x^* = -\left(\frac{\beta\tau}{\gamma}\right)^{-\frac{1}{\gamma+1}},$$

and that from (A.7), we have

$$(-t) = \left(\frac{\tau}{\gamma\beta^\gamma}\right)^{-\frac{1}{\gamma+1}}. \qquad \text{(A. 18)}$$

In a similar way we can obtain the two-sided Laplace formula expression for denominator of $B(u)$ in (2.11), defined by $D(u)$, putting $\tau = \frac{u^2}{2}$, such that

$$D(u) = \int_{-\infty}^{\infty} e^{\tau t}\, \theta_2(t)\, dt,$$

where $\theta_2(t) = 0$ when $t \geq 0$ and when $t < 0$,

$$\theta_2(t) = (-t)^{-1} g_0(-t^{-1}) \qquad \text{(A. 19)}$$

$$= (-t)^{-1} \frac{\gamma(-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\beta^\gamma} e^{-(-t/\beta)^{-\gamma}}$$

$$(\alpha > 0,\ \beta > 0,\ \gamma > 0).$$

Hence we can set

$$\theta_2(t) = h_2(t)\, e^{-\psi(t)}, \qquad \text{(A. 20)}$$

$$\text{where } h_2(t) = (-t)^{-1} \frac{\gamma(-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\beta^\gamma}$$

$$\text{and } \psi(t) = (-t/\beta)^{-\gamma} \qquad (t < 0).$$

Thus from (A. 6) we also have

$$D(u) = \int_{-\infty}^{\infty} e^{\tau t}\,\theta_2(t)\,dt \sim \beta_2(\tau)\,e^{\psi^*(\tau)}\,, \tag{A. 21}$$

where

$$\beta_2(\tau) = \sqrt{2}\,s(t)\,h_2(t)$$

$$= \sqrt{2}\left\{\frac{1}{\sqrt{\beta^\gamma\,\gamma\,(1+\gamma)}}\,(-t)^{1+\gamma/2}\right\}\left\{(-t)^{-1}\,\frac{\gamma\,(-t)^{1-\alpha}}{\Gamma(\alpha/\gamma)\,\beta^\gamma}\right\}$$

$$= \frac{\sqrt{2}\,\gamma^2\,(1+\gamma)}{(\beta^\gamma\,\gamma\,(1+\gamma))^{3/2}\,\Gamma[\frac{\alpha}{\gamma}]}\,(-t)^{\frac{1}{2}\,(2-2\,\alpha+\gamma)}, \tag{A. 22}$$

and $(-t) = \left(\frac{\tau}{\gamma\,\beta^\gamma}\right)^{-\frac{1}{\gamma+1}}$.

Therefore, from (A.15), (A.18) and (A.22) and by (2.11) we have, as $u\to\infty$,

$$B(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{f(u)}{u\;D(u)}\right)$$

$$\sim \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{\beta_1(\tau)}{u\;\beta_2(\tau)}\right) \tag{A. 23}$$

$$= \frac{\sqrt{2\pi}\;u^{-1}}{\sqrt{-\rho^{(2)}(0)}}\,(-t)^{-\frac{1}{2}}$$

$$= \frac{\sqrt{2\pi}\;u^{-1}}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{u^2}{2\,\gamma\,\beta^\gamma}\right)^{\frac{1}{2(\gamma+1)}}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-\rho^{(2)}(0)}}\left(\frac{1}{2\,\gamma\,\beta^\gamma}\right)^{\frac{1}{2(\gamma+1)}}\,u^{-\frac{\gamma}{\gamma+1}}\,,$$

since $\tau = u^2/2$. This completes the proof of Theorem.

Conclusion

We have provided a proof of Theorem for the asymptotic order of the expected mean length of excursions for the processes with a scale mixture of normal (SMN) distribution whose mixing distribution is the generalized Gamma distribution, which was the conjecture presented in the previous paper of Tanaka [15].

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