Asymptotic order of the expected length of excursions for the processes with a Scale mixture of normal distribution

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Abstract. The paper treats of the asymptotic order of the expected mean length of excursions for some ellipsoidal processes with a scale mixture of normal (SMN) distribution when the level is sufficiently large, which is the limiting values of ratios of two functions. It is shown that the Abelian theorem for the Laplace transforms and the L'Hopital's rule are useful to estimate the asymptotic order. A general condition to derive the true asymptotic order of the ratios is also discussed.

Keywords: *L'Hopital's rule, length of excursions, Scale mixture of normal distribution, Laplace transform, Abelian theorem*

1. Introduction

It is known that for the zero-mean stationary Gaussian process X_t with autocovariance γ (h) in continuous time the order of the expected mean length of excursions above level u is given by $O(u^{-1})$ when u is sufficiently large (see, for example, Kedem [8], p.138). It is also seen that in the process having the Pearson Type VII distribution its order of the length of excursions in terms of u is $O(u^0)$, that is, constant in u (Tanaka and Shimizu [9]). There is a paper which extended the results to other stationary ellipsoidal processes which have the Laplace distribution, the generalized Laplace distribution and the Logistic distributions (for example, Tanaka and Shimizu [11]).

Following the previous paper of Tanaka [10], we shall suppose throughout that {X(t)} is a stationary zero-mean and unit-variance ellipsoidal process with the probability density function f(x) and the autocorrelation function ρ(h) which is twice differentiable at h = 0. An expected mean length of excursions above level u discussed in Tanaka [10], and Tanaka and Shimizu ([9],[11]) for the discrete time ellipsoidal process is the following ratio of the two integrals:

$$I_u(\mathrm{N}) = \frac{\int_u^\infty f(x)\,dx}{\frac{1}{\pi\Delta}\int_\rho^1 \frac{1}{\sqrt{1-t^2}}\int_0^\infty f\left(\sqrt{x^2+\frac{2u^2}{1+t}}\right)dx\,dt}. \qquad (1.1)$$

The continuous time formula of (1.1) is also given as

$$\lim_{N\to\infty} I_u(N) = \frac{\pi\int_u^\infty f(x)\,dx}{\sqrt{-\rho^{(2)}(0)}\int_0^\infty f\left(\sqrt{x^2+u^2}\right)dx} \qquad (1.2)$$

$$= A(u), \quad say.$$

Particularly when f(x) is a standard normal N(0,1) density and Φ is the distribution function, we have

$$A(u) = \frac{2\pi\,[1-\Phi(u)]}{\sqrt{-\rho^{(2)}(0)}\,exp\{-\frac{1}{2}u^2\}} \qquad (1.3)$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\,\frac{1}{u} \qquad as\ u\to\infty.$$

(see (a) in Problem 15 of Kedem [8]). Then it is interesting to estimate the order of smallness of A(u) when u→∞ for ellipsoidal processes with a non-Gaussian distribution function f(x).

Let $H(u)$ be a power function of u with a negative order, i.e. $H(u) = C_0\,u^{-\alpha}$ for $\alpha > 0$, $C_0 > 0$. If a function G(u) ~ $H(u)$ as u → ∞, then $H(u)$ is called the asymptotic order function of G(u) with the order (- α) when u →∞. For example, it is seen from (1.3) that the asymptotic order function of A(u) for the Gaussian process is

$$H(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\ \frac{1}{u} \tag{1.4}$$

and its order is *(-1)*.

To consider the limit of the ratio in (1.2) when u→∞, we introduce the completely monotonic functions, because almost ellipsoidal density functions are completely monotonic (see Andrews and Mallows [1]). We say that the function f(x) is completely monotonic in [0,∞) if it satisfies $(-1)^k f^{(k)}(x) \geq 0$ for $0 \leq x < \infty$. Bernstein's theorem (see Widder [13], Theorem 19-b) shows that if f(x) is completely monotonic, f(x) is expressed as the Laplace transform of some function α(t) such that

$$f(s) = \int_0^\infty e^{-st}\, d\,\alpha(t)$$

$$= \int_0^\infty e^{-st}\, \theta(t)\, dt, \tag{1.5}$$

where α(t) is bounded and non-decreasing in [0, ∞) and absolutely continuous, i.e. $d\,\alpha(t) = \theta(t)\,dt$. Furthermore the density function f(s) in (1.5) can be also expressed by a scale mixture of normal distribution (SMN) with a mixing function G(a), or Mellin-Stieltjes transform of G(a) with the kernel N(0,1), i.e.

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi a}}\, e^{-\frac{x^2}{2a}}\, dG(a)$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi a}}\, e^{-\frac{x^2}{2a}}\, g(a)\, da, \tag{1.6}$$

where $s = \frac{x^2}{2}$ and G(a) is an absolutely continuous and $d\,G(a) = g(a)\,da$. From (1.5) and (1.6) we have

$$\theta(t) = \frac{t^{-3/2}}{\sqrt{2\pi}}\, g\left(\frac{1}{t}\right). \tag{1.7}$$

For example, in the case of the Gaussian process (when f(s) is the density of standard normal distribution), we have G(a) = U(a-1) = 0 (a ≤ 1), = 1 (a > 1), and then g(a) = $\delta(a-1)$ (*Dirac's delta* function).

2. Results

Using the well-known L'Hospital's rule (see Hardy [6] and Tichmarsh [12]), we can derive the following lemma.

Lemma A. Let p(x), q(x) and r(x) be the real-valued continuous and differentiable functions on some neighborhood [a, ∞) of infinity. Suppose that r(x) q(x) ≠ 0, q' (x) ≠ 0 for x ∈ [a, ∞), and $\lim_{x\to\infty} p(x) = 0$, $\lim_{x\to\infty} r(x)\, q(x) = 0$.

If (i) there exists constant $C_1 > 0$ such that $\lim_{x\to\infty} \dfrac{p'(x)}{r(x)\, q'(x)} = C_1$,

$$\text{i.e.} \quad \frac{p'(x)}{q'(x)} \sim C_1\, r(x) \quad \text{as } x \to \infty,$$

(ii) $\lim_{x\to\infty} \dfrac{r'(x)\, q(x)}{r(x)\, q'(x)} = C_3$, where C_3 is a constant,

then there exists a constant $C_2 > 0$ such that $\lim_{x\to\infty} \dfrac{p(x)}{r(x)\, q(x)} = C_2$,

$$\text{i.e.} \quad \frac{p(x)}{q(x)} \sim C_2\, r(x) \quad \text{as } x \to \infty,$$

where $C_2 = \dfrac{C_1}{1+C_3}$. In addition, if $C_3 = 0$ in the condition (ii), then

$$\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)} \quad \text{as } x \to \infty.$$

Corollary A. Suppose that $r(x) = x^{\alpha}$ ($\alpha \neq 0$) in Lemma A.
If q(x) satisfies the condition (i) and

(ii)' $\lim_{x\to\infty} \dfrac{q(x)}{x\, q'(x)} = 0$, then

$$\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)} \quad \text{as } x \to \infty.$$

Note that if $r(x) = x^{\alpha}$ $(\alpha = 0)$, the condition (ii) of Lemma 1 is always holds.

Applying Lemma A (Corollary A) to the ratio A(u) of (1.2), the partial differentiation with respect to u and taking the limit as u →∞ will lead the following result.

Theorem 1. Suppose that f(x) is a scale mixture of normal distribution (SMN) with a mixing density function g(a) and it is expressed by (1.6). Let A(u) be the expected mean length of excursions above a level u in (1.2) such that, for some $\alpha \geq 0$,

$$A(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \left(\frac{\int_u^{\infty} f(x)\, dx}{\int_0^{\infty} e^{-\frac{u^2}{2a}} g(a)\, da} \right) = O(u^{-\alpha}) \qquad (2.1)$$

as u→∞. If $\alpha > 0$ and g(a) satisfies either

$$\lim_{u \to \infty} \left(\frac{\int_0^{\infty} e^{-\frac{u^2}{2a}} g(a)\, da}{u^2 \int_0^{\infty} \frac{1}{a} e^{-\frac{u^2}{2a}} g(a)\, da} \right) = 0, \qquad (2.2)$$

or

$$\lim_{u \to \infty} \left(\frac{\int_u^{\infty} f(x)\, dx}{u\, f(u)} \right) = 0, \qquad (2.2)'$$

then, as u →∞,

$$A(u) \sim \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \left(\frac{f(x)}{u \int_0^{\infty} \frac{1}{a} e^{-\frac{u^2}{2a}} g(a)\, da} \right) \qquad (2.3)$$

$$= B(u)\ ,\ say.$$

Furthermore if A(u) = O(1), then (2.3) always holds.

We should note that these results are modification of the previous results given by the author (Lemma 1 and Proposition in Tanaka [10]). The ratio B(u) in (2.3) can be expressed in terms of the Laplace transforms such that

$$B(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \left(\frac{\int_0^{\infty} e^{-\left(\frac{u^2}{2}\right)t}\, \theta_1(t)\, dt}{u \int_0^{\infty} e^{-\left(\frac{u^2}{2}\right)t}\, \theta_2(t)\, dt} \right), \tag{2.4}$$

where $\theta_1(t) = \frac{t^{-3/2}}{\sqrt{2\pi}}\, g(t^{-1})$ and $\theta_2(t) = t^{-1}\, g(t^{-1})$.

From the Laplace transform formula (2.4) we may evaluate the asymptotic order of A(u) of the scale mixture of normal distribution f(x) by using the well-known Abelian theorem for the Laplace transform (see, for example, Bingham, Goldie and Teugels [2], Widder [13]). When f(x) is a normal density function, B(u) is simpler than A(u) and it is easily seen that B(u) is of order 1/u, see Example 1 below. We can also use Theorem 1 for the evaluations of the asymptotic order of A(u) for some non Gaussian distribution, such as the generalized Laplace distribution, the Pearson Type VII distribution and the Logistic distribution and so on. Some of these are presented by the following Examples 1-4. A new result will be also given in Example 5 for the inverse Gauss distribution as a mixing function of the *SMN (scale mixture of normal)*. Example 6 will give a conjecture for the case of the function f(x) with a generalized Gamma distribution as the mixing density function, which is an extension of the case for the generalized Laplace distribution.

Example 1. Let the process X_t have the *standard normal* density function :

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}. \tag{2.5}$$

Then the B (u) in (2.3) with g (a) = δ (a − 1) (Dirac's delta function) is given by

$$B(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}}{u \int_0^{\infty} \frac{1}{a}\, e^{-\frac{u^2}{2a}}\, g(a)\, da}$$

$$= \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \; \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}{u \; e^{-\frac{u^2}{2}}}$$

$$= \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \; u^{-1}.$$

We should note that in this case it is easy to show the condition (2.2) holds.

Example 2. Let the process X_t have the *generalized Laplace distribution* with parameters $\gamma > 1/4$ and $\sigma > 0$:

$$f(x) = \frac{\sqrt{\pi}\,\sigma}{\text{Gamma}(\gamma)} \left(\frac{x}{2\sigma}\right)^{\gamma-\frac{1}{2}} \text{BesselK}\left(\gamma - \tfrac{1}{2}, \tfrac{x}{\sigma}\right), \tag{2.6}$$

where BesselK(,) is the modified Bessel function of the third kind (see Johnson, Kotz and Balakrishnan [7], Tanaka and Shimizu [9]). The asymptotic order function of A(u) of (1.2) when $u \to \infty$ is evaluated by use of B(u) in (2.3) such that

$$B(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \sqrt{\sigma}\, u^{-1/2} \frac{\text{BesselK}[\gamma-1/2,\, u/\sigma]}{\text{BesselK}[\gamma-1,\, u/\sigma]} \tag{2.7}$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \sqrt{\sigma}\, u^{-1/2} \quad \text{as } u \to \infty.$$

For the mixing function g(a) of f(x) is the density of the Gamma distribution,

$$g(a) = \frac{a^{\gamma-1}}{\text{Gamma}[\gamma]\,(2\sigma^2)^{\gamma}} e^{-\left(\frac{a}{2\sigma^2}\right)}, \tag{2.8}$$

and then

$$\int_0^{\infty} e^{-\frac{u^2}{2a}} g(a)\, da = \frac{2^{1-\frac{\gamma}{2}} \left(\frac{2\sigma^2}{u^2}\right)^{-\gamma/2} \text{BesselK}[\gamma,\, u/\sigma]}{\text{Gamma}[\gamma]},$$

$$\int_0^\infty \frac{1}{a}\, e^{-\frac{u^2}{2a}}\, g(a)\, da = \frac{2\,(2\,\sigma^2)^{-\gamma}\,(\sigma\, u)^{\gamma-1}\ \mathrm{BesselK}[\gamma-1,\ u/\sigma]}{\mathrm{Gamma}[\gamma]}.$$

The condition (2.2) holds. In this case we have the Laplace transform formulas in (2.4) with

$$\theta_1(t) = \frac{(2\,\sigma^2)^{-\gamma}}{\sqrt{2\,\pi}\ Gamma[\gamma]}\ t^{-(\gamma+1/2)}\ e^{-\frac{t^{-1}}{2\,\sigma^2}},$$

$$\theta_2(t) = \frac{(2\,\sigma^2)^{-\gamma}}{Gamma[\gamma]}\ t^{-\gamma}\ e^{-\frac{t^{-1}}{2\,\sigma^2}}.$$

Then the Abelian theorem of the Laplace transform (see Tanaka and Shimizu [11]) shows that

$$\frac{\theta_1\,(t)}{\theta_2\,(t)} \sim \frac{t^{-(1/2)}}{\sqrt{2\,\pi}} \qquad as\ t \to 0+$$

implies

$$\frac{\int_0^\infty e^{-s\,t}\,\theta_1(t)\,dt}{\int_0^\infty e^{-s\,t}\,\theta_2(t)\,dt} \sim \frac{(2\,\sigma^2\,s)^{1/4}}{\sqrt{2\,\pi}} = \frac{(\sigma\,u)^{1/2}}{\sqrt{2\,\pi}} \qquad as\ u \to \infty.$$

Therefore we have $B(u) \sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\ \sqrt{\sigma}\ u^{-1/2}$ as $u \to \infty$.

Example 3. Let the process X_t have the *Pearson Type VII* distribution :

$$f\,(x) = \frac{A^{2\,\nu}}{\sqrt{\pi}}\ \frac{Gamma[\nu+1/2]}{Gamma[\nu]}\ (A^2 + x^2)^{-(\nu+1/2)} \qquad (2.9)$$

for $\nu > 0$, $A > 0$. Then the asymptotic order of A(u) when u → ∞ is given by

$$H\,(u) = \sqrt{\frac{2\,\pi}{-\rho^{(2)}\,(0)}}\ \frac{Gamma\,[\nu+1/2]}{\sqrt{2}\ Gamma\,[\nu+1]}. \qquad (2.10)$$

(see Tanaka and Shimizu [9]). In this case the condition (2.2) also hold, since the mixing function g(a) of f(x) is the density of the inverse Gamma distribution,

$$g(a) = \frac{A^{2\nu}\, a^{-(\nu+1)}}{2^{\nu}\, Gamma[\nu]}\, e^{-\frac{A^2}{2a}}, \tag{2.11}$$

and then

$$\int_0^\infty e^{-\frac{u^2}{2a}}\, g(a)\, da = A^{2\nu}\,(A^2+u^2)^{-\nu},$$

$$\int_0^\infty \frac{1}{a}\, e^{-\frac{u^2}{2a}}\, g(a)\, da = 2\, A^{2\nu}\,(A^2+u^2)^{-1-\nu}\, \frac{Gamma[1+\nu]}{Gamma[\nu]}.$$

Hence

$$B(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\; \frac{(A^2+u^2)^{\frac{1}{2}}\, Gamma[\frac{1}{2}+\nu]}{u^2\,\sqrt{\pi}\; Gamma[1+\nu]} \tag{2.12}$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\; \frac{Gamma[\frac{1}{2}+\nu]}{\sqrt{2}\; Gamma[1+\nu]} \qquad as\ u \to \infty.$$

Thus the order is a constant, and the condition (2.2) always holds. In this case the Laplace transform formulas in (2.4) are given by

$$\theta_1(t) = \frac{2^{-\frac{1}{2}-\nu}\, A^{2\nu}\, e^{-\frac{A^2 t}{2}}\, t^{\nu-\frac{1}{2}}}{\sqrt{\pi}\; Gamma[\nu]} \sim \frac{2^{-\frac{1}{2}-\nu}\, A^{2\nu}}{\sqrt{\pi}\; \mathrm{Gamma}[\nu]}\, t^{-\frac{1}{2}+\nu},$$

$$\theta_2(t) = \frac{2^{-\nu}\, A^{2\nu}\, e^{-\frac{A^2 t}{2}}\, t^{\nu}}{\mathrm{Gamma}[\nu]} \sim \frac{2^{-\nu}\, A^{2\nu}}{\mathrm{Gamma}[\nu]}\, t^{\nu} \qquad \text{as } t \to 0.$$

Since these functions $\theta_1(t)$ and $\theta_2(t)$ are regularly varying (see Bingham, Goldie and Teugels [2]), the Abelian theorem for the Laplace transform of these functions shows

$$\frac{\int_0^\infty e^{-st}\,\theta_1(t)\,dt}{\int_0^\infty e^{-st}\,\theta_2(t)\,dt} \sim \frac{(s)^{1/2}}{\sqrt{2\pi}}\; \frac{Gamma[\frac{1}{2}+\nu]}{Gamma[1+\nu]}$$

$$= \frac{\left(\frac{u^2}{2}\right)^{1/2}}{\sqrt{2\pi}}\; \frac{Gamma[\frac{1}{2}+\nu]}{Gamma[1+\nu]} \quad as\ u \to \infty.$$

Therefore we have $B(u) \sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \frac{Gamma[\frac{1}{2}+\nu]}{Gamma[1+\nu]}$ as $u \to \infty$.

Example 4. Let the process X_t have the *Logistic* distribution with the density

$$f(x) = \frac{e^x}{(1+e^x)^2}. \tag{2.13}$$

Then the asymptotic order of A(u) when u → ∞ is given by

$$H(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}} \frac{1}{\sqrt{u}}. \tag{2.14}$$

In this case the mixing function g(a) of f(x) is given by

$$g(a) = \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2}{2} a} \qquad (a > 0). \tag{2.15}$$

Then

$$\int_0^{\infty} \frac{1}{a} e^{-\frac{u^2}{2a}} g(a)\, da = 2 \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \,\mathrm{BesselK}[0, k\, u]$$

$$\sim -\frac{\sqrt{2\pi}\,\mathrm{PolyLog}[-\frac{3}{2}, -e^{-u}]}{\sqrt{u}}$$

$$\sim \frac{\sqrt{2\pi}\, e^{-u}}{\sqrt{u}} \qquad as \;\; u \to \infty.$$

Hence

$$B(u) \sim \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \frac{\frac{e^u}{(1+e^u)^2}}{u\, \left(\frac{\sqrt{2\pi}\, e^{-u}}{\sqrt{u}}\right)} \tag{2.16}$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\, u^{-1/2} \qquad as \;\; u \to \infty.$$

In this case the Laplace transform formulas in (2.4) are given by

$$\theta_1(t) = \frac{2^{-\frac{1}{2}-\nu} A^{2\nu} e^{-\frac{A^2 t}{2}} t^{\nu-\frac{1}{2}}}{\sqrt{\pi}\ Gamma[\nu]} \sim \frac{2^{-\frac{1}{2}-\nu} A^{2\nu}}{\sqrt{\pi}\ \mathrm{Gamma}[\nu]} t^{-\frac{1}{2}+\nu},$$

$$\theta_2(t) = \frac{2^{-\nu} A^{2\nu} e^{-\frac{A^2 t}{2}} t^{\nu}}{\mathrm{Gamma}[\nu]} \sim \frac{2^{-\nu} A^{2\nu}}{\mathrm{Gamma}[\nu]} t^{\nu} \qquad \text{as } t \to 0,$$

(see Tanaka and Shimizu [11]). Similar to the Example 3, since the functions $\theta_1(t)$ and $\theta_2(t)$ are also regularly varying, we have

$$\frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \frac{\int_0^\infty e^{-st}\,\theta_1(t)\,dt}{u\int_0^\infty e^{-st}\,\theta_2(t)\,dt} \sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\, u^{-1/2} \qquad as\ u \to \infty.$$

Example 5. Let the process X_t have the *SMN* (*scale mixture of normal*) distribution f (x) with the mixing density function, for a > 0, $\mu > 0$ and $\lambda > 0$,

$$g(a) = \sqrt{\frac{\lambda}{2\pi a^2}}\, e^{-\lambda \frac{(a-\mu)^2}{2\mu^2 a}}, \tag{2.17}$$

which is the inverse Gauss density function, then we have

$$f(x) = \frac{e^{-\sqrt{x^2+\lambda}\sqrt{\frac{\lambda}{\mu^2}}+\frac{\lambda}{\mu}}\sqrt{2\pi}\sqrt{\lambda}}{2\pi\sqrt{x^2+\lambda}}. \tag{2. 18}$$

Then

$$B(u) = \frac{2\pi}{\sqrt{-\rho^{(2)}(0)}} \frac{\dfrac{e^{-\sqrt{u^2+\lambda}\sqrt{\frac{\lambda}{\mu^2}}+\frac{\lambda}{\mu}}\sqrt{2\pi}\sqrt{\lambda}}{2\pi\sqrt{u^2+\lambda}}}{u\left(\dfrac{2e^{\lambda/\mu}\sqrt{\lambda}\sqrt{\frac{\lambda}{\mu^2}}\ \mathrm{BesselK}[1,\sqrt{u^2+\lambda}\sqrt{\frac{\lambda}{\mu^2}}]}{\sqrt{2\pi}\sqrt{u^2+\lambda}}\right)}$$

$$\sim \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\,(\mu^{-1/2}\lambda^{-1/4})\,u^{-1/2} \qquad \text{as } u \to \infty. \tag{2.19}$$

Hence the asymptotic order function of A (u) when u $\to\infty$ is given by

$$H(u) = \sqrt{\frac{2\pi}{-\rho^{(2)}(0)}}\,(\mu^{-1/2}\lambda^{-1/4})\,u^{-1/2}. \tag{2.20}$$

Example 6 (a conjecture). Let the process X_t have the *SMN (scale mixture of normal)* distribution f(x) with the mixing density function,

$$g(a) = \frac{\gamma\, a^{\alpha-1}}{\text{Gamma}[\alpha/\gamma]\,\beta^{\alpha}}\, e^{-(a/\beta)^{\gamma}} \quad (\alpha>0,\ \beta>0,\ \gamma>0). \tag{2.21}$$

(the density of the *Generalized Gamma* distribution, see Johnson, Kotz and Balakrishnan [7], which is a Gamma distribution when $\gamma=1$).

The mixing density of the Pearson Type VII distribution may be asymptotically equivalent to that of (2.15) when $\gamma\to 0$. The generalized Laplace, logistic and the density function (2.13) in Example 5 also have the same kind mixing function of (2.15) with $\gamma = 1$. When $\gamma\to\infty$, the mixing function of (2.15) will be δ(a-1) (Dirac's delta function) and then this is the mixing function of the standard normal distribution.

Therefore the results of Examples 1~5 above will lead to a following conjecture:

If the mixing density function g(s) is a generalized Gamma distribution of (2.10), the asymptotic order function H(u) of A(u) in (1.2) depends only on the parameter γ, and it will be given by

$$H(u) = O\left(u^{-\frac{\gamma}{1+\gamma}}\right). \tag{2.22}$$

Examples 1-5 above will be special cases of the result (2.22):

when $\gamma\to 0$ (from Example 2; Pearson Type VII),

$$\text{H(u)} = O(u^{-0}) = O(1);$$

when $\gamma = 1$ (from Examples 3-5; generalized Laplace, logistic),

$$\text{H(u)} = O(u^{-1/2});$$

when $\gamma \to \infty$ (from Example 1; Normal distribution),

$$H(u) = O(u^{-1}).$$

Unfortunately until now we can not prove the conjecture directly because the Laplace transforms in (2.3) are not able to be evaluated expressively. Then in the next section we shall illustrate the following three cases of γ :

(1) for $\gamma = 1/2$, $H(u) = O(u^{-1/3})$,

(2) for $\gamma = 1$, $H(u) = O(u^{-1/2})$ (this is the only known case)

(3) for $\gamma = 2$, $H(u) = O(u^{-2/3})$.

3. Illustrations

(1) Let $\gamma = 1/2$ in (2.21), then the ratios A(u) and B(u) in Theorem 1 may be given by the following forms:

$$A(u) = \frac{\text{MeijerG}[\{\{\},\{\}\},\{\{0,\frac{1}{2},\frac{3}{2}\},\{\}\},\frac{u^2}{8}]}{2\sqrt{\pi}\ \text{MeijerG}[\{\{\},\{\}\},\{\{0,1,\frac{3}{2}\},\{\}\},\frac{u^2}{8}]}, \tag{2.23}$$

$$B(u) = \frac{\sqrt{\frac{2}{\pi}}\ \text{MeijerG}[\{\{\},\{\}\},\{\{0,\frac{1}{2},1\},\{\}\},\frac{u^2}{8}]}{u\,\text{MeijerG}[\{\{\},\{\}\},\{\{0,0,\frac{1}{2}\},\{\}\},\frac{u^2}{8}]}, \tag{2.24}$$

where MeijerG[.] is a Meijer's G function (see Gradshteyn and Ryzhik [5]).

Figure 1 and Figure 2 show that the asymptotic order functions of A(u) and B(u) seem to be of same order as O($u^{-1/3}$), which agrees with the conjecture.

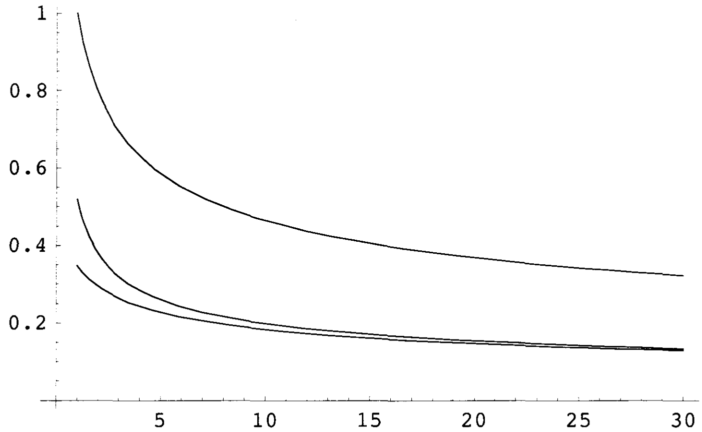

Figure 1. Graphs of A(u), B(u) and $u^{-1/3}$(upper line).

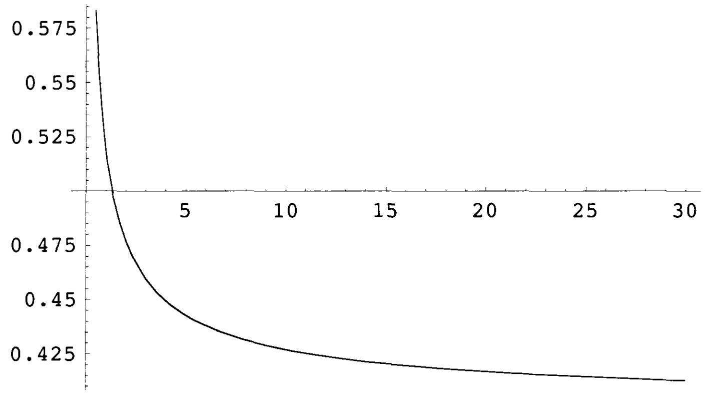

Figure 2. The graph of the ratio { B(u) / $u^{-1/3}$},
which seems to tend to a constant as u $\rightarrow \infty$.

(2) Let $\gamma = 1$ in (2.21), then we have

$$\mathrm{A(u)} = \frac{e^{-\sqrt{2}\,x}}{2\sqrt{2}\,x\,\mathrm{BesselK}\left[1,\sqrt{2}\,x\right]}, \tag{2.25}$$

$$\mathrm{B(u)} = \frac{e^{-\sqrt{2}\,x}}{2\sqrt{2}\,x\,\mathrm{BesselK}\left[0,\sqrt{2}\,x\right]}. \tag{2.26}$$

It is easily seen that A(u) and B(u) are O($u^{-1/2}$) as u $\to \infty$.

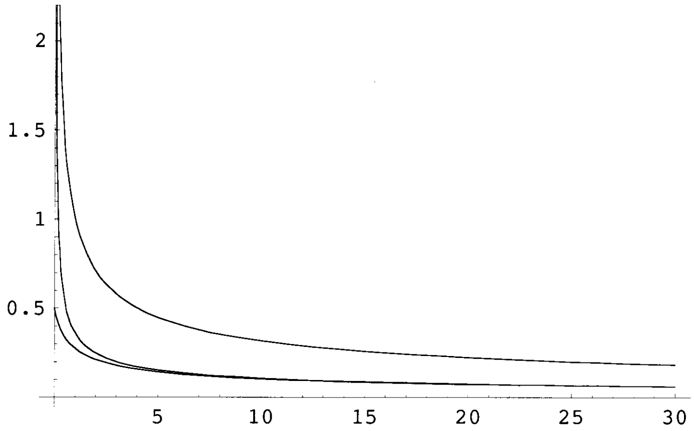

Figure 3. Graphs of A(u), B(u) and $u^{-1/2}$(upper line).

(3) Let $\gamma = 2$ in (2.21), then we have

$$A(u) = 2\Big\{3\pi + 6\sqrt{\pi}\, x^2\, \text{HypergeometricPFQ}\Big[\{\tfrac{1}{2}\}, \{\tfrac{3}{4}, \tfrac{5}{4}, \tfrac{3}{2}\}, -\tfrac{x^4}{16}\Big] - 2\sqrt{2}\, x\Big(6\,\text{Gamma}[\tfrac{5}{4}]\,\text{HypergeometricPFQ}\Big[\{\tfrac{1}{4}\}, \{\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}\}, -\tfrac{x^4}{16}\Big] + x^2\,\text{Gamma}[\tfrac{3}{4}]\,\text{HypergeometricPFQ}\Big[\{\tfrac{3}{4}\}, \{\tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}\}, -\tfrac{x^4}{16}\Big]\Big)\Big\} \Big/ \Big(3x^2\,\text{MeijerG}\Big[\{\{\}, \{\}\}, \{\{-\tfrac{1}{2}, 0, 0\}, \{\}\}, \tfrac{x^4}{16}\Big]\Big), \tag{2.27}$$

$$B(u) = \sqrt{2}\,\Big(2\,\text{Gamma}[\tfrac{5}{4}]\,\text{HypergeometricPFQ}\Big[\{\}, \{\tfrac{1}{2}, \tfrac{3}{4}\}, -\tfrac{x^4}{16}\Big] - \sqrt{2\pi}\,\sqrt{x^2}\,\text{HypergeometricPFQ}\Big[\{\}, \{\tfrac{3}{4}, \tfrac{5}{4}\}, -\tfrac{x^4}{16}\Big] + x^2\,\text{Gamma}[\tfrac{3}{4}]\,\text{HypergeometricPFQ}\Big[\{\}, \{\tfrac{5}{4}, \tfrac{3}{2}\}, -\tfrac{x^4}{16}\Big]\Big) \Big/ \Big(x\,\text{MeijerG}\Big[\{\{\}, \{\}\}, \{\{0, 0, \tfrac{1}{2}\}, \{\}\}, \tfrac{x^4}{16}\Big]\Big). \tag{2.28}$$

where HypergeometricPFQ[.] is the generalized hypergeometric function ${}_pF_q$ (see Gradshteyn and Ryzhik [5]).

Figure 4 bellow shows that the asymptotic order functions of A(u) and B(u) seem to be of same order as O($u^{-2/3}$), which also agrees with the conjecture.

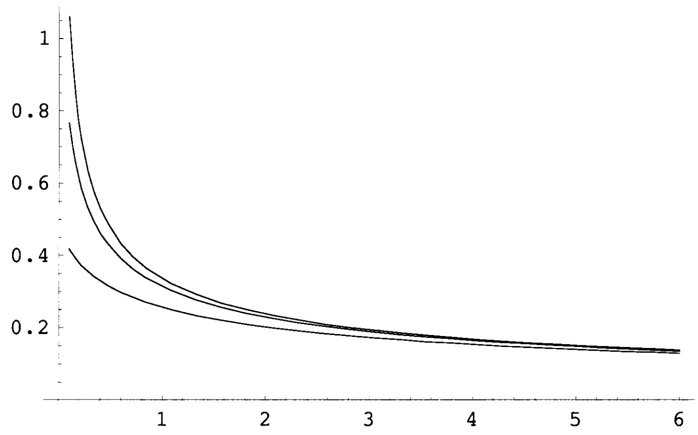

Figure 4. Graphs of A(u), B(u) and $u^{-2/3}$(lowest line).

Appendix

Proof of Lemma A

$$\frac{p'(x)}{(r(x)\,q(x))'} = \frac{p'(x)}{r'(x)\,q(x) + r(x)\,q'(x)}$$

$$= \frac{p'(x)}{r(x)\,q'(x)\left[\frac{r'(x)\,q(x)}{r(x)\,q'(x)} + 1\right]}$$

$$= \frac{p'(x)}{r(x)\,q'(x)}\left[1 + \frac{r'(x)\,q(x)}{r(x)\,q'(x)}\right]^{-1},$$

Therefore if the conditions (i) and (ii) hold, applying the L'Hopital's rule, we have

$$Limit_{x\to\infty}\frac{p(x)}{r(x)\,q(x)} = Limit_{x\to\infty}\frac{p'(x)}{(r(x)\,q(x))'}$$

$$= \ Limit_{x\to\infty} \left\{ \frac{p\,'(x)}{r(x)\,q\,'(x)} \left[1 + \frac{r\,'(x)\,q(x)}{r(x)\,q\,'(x)} \right]^{-1} \right\}$$

$$= \ Limit_{x\to\infty} \left\{ \frac{p\,'(x)}{r(x)\,q\,'(x)} \right\} Limit_{x\to\infty} \left\{ \left[1 + \frac{r\,'(x)\,q\,(x)}{r(x)\,q\,'(x)} \right]^{-1} \right\}$$

$$= \ \frac{C_1}{1 + C_3}$$

Hence we have a constant $C_2 = \dfrac{C_1}{1 + C_3} > 0$ such that

$$\frac{p(x)}{q(x)} \sim C_2\, r(x) \qquad as\ x \to \infty.$$

Furthermore if $C_3 = 0$, we have $C_2 = C_1$, and the converse also holds. □

Proof of Corollary A

Suppose $r(x) = x^{\alpha}$, then $\dfrac{r\,'(x)}{r(x)} = \alpha\, x^{-1}$. Hence the condition (ii) in Lemma A with $C_3 = 0$ is given by

$$\lim_{x\to\infty} \frac{\alpha\ q(x)}{x\ q\,'(x)} = 0.$$

Thus Corollary A is obtained. □

We shall next consider some special cases of the function q(x) in Corollary A.

(1) *Suppose* $q(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ *(Normal distribution).*

Let $p(x) = 1 - \Phi(x) = \int_x^{\infty} q(t)\, dt$, then it is well-known that

$$\frac{p(x)}{q(x)} \sim x^{-1} \quad \text{as } x \to \infty.$$

On the other hand, $p'(x) = -\,q(x)$ and $q\,'(x) = (-x)\,q(x)$. Thus

$$\frac{p'(x)}{q'(x)} \sim x^{-1} \quad \text{as } x \to \infty.$$

Hence $\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)}$ (as $x\to\infty$) as $x\to\infty$. In this case the condition (ii) holds, since

$$\frac{q(x)}{x\,q'(x)} = -x^{-2} \to 0 \text{ as } x \to \infty.$$

(2) Suppose $q(x) = e^{\frac{1}{x}}$ and $p(x) = x^{-\alpha}e^{\frac{1}{x}}$. Then we have

$$q'(x) = (-x^{-2})q(x),$$

$$\frac{q(x)}{x\,q'(x)} = -x \to -\infty = C_3 \text{ as } x \to \infty.$$

Hence the condition (ii) does not hold. In fact $\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)}$ (as $x\to\infty$) does not hold, since $\frac{p(x)}{q(x)} = x^{-\alpha}$ and

$$\frac{p'(x)}{q'(x)} = \frac{(-x^{-\alpha-2})\,q(x)\,(\alpha x+1)}{(-x^{-2})\,q(x)} \sim x^{-\alpha+1} \quad \text{as } x \to \infty.$$

(3) Suppose $q(x) = x^{-a}$ $(a > 0)$ and $p(x) = x^{-\alpha}q(x)$ $(\alpha > 0)$. Then we have

$$q'(x) = (-a)\,x^{-a-1} = (-a)\,x^{-1}q(x)$$

$$\frac{q(x)}{x\,q'(x)} = \frac{1}{(-a)} = C_3 \neq 0.$$

$$\frac{p'(x)}{q'(x)} = \frac{(-\alpha x^{-\alpha-1})\,q(x) + x^{-\alpha}\,(-a)\,x^{-1}\,q(x)}{(-ax^{-1})\,q(x)} = \frac{(\alpha+a)\,x^{-\alpha}}{a} \neq x^{-\alpha}.$$

Hence $\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)}$ (as $x\to\infty$) does not hold.

(4) Suppose $q(x) = e^{-h(x)}$ $(h(x) > 0, h'(x) > 0)$ and $p(x) = x^{-\alpha} q(x)$ $(\alpha > 0)$.

Then we have $q'(x) = (-h'(x))\, q(x)$, and

$$\frac{q(x)}{x\, q'(x)} = \frac{1}{(-x)\, h'(x)}.$$

Thus the condition (ii) holds iff $\text{Limit}_{x\to\infty}\, [x\, h'(x)] \neq 0$,

and also we have

$$\frac{p(x)}{q(x)} \sim \frac{p'(x)}{q'(x)} \quad (\text{as } x\to\infty) \quad \text{iff} \quad \text{Limit}_{x\to\infty}\, [x\, h'(x)] = \infty.$$

Specially if $h(x) = x^a$ $(a > 0)$, the condition (ii) holds, but if $h(x) = Log(x)$, the condition (ii) does not holds.

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